ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ UNIVERSITY OF IOANNINA

ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ DEPARTMENT OF MATHEMATICS

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On the Dual Space for the Strict Topology β_1 and the Space $M(X)$ in Function Spaces

J Aguayo, A. K. Katsaras and S. Navarro

Abstract

Let $C_b(X)$ be the space of all bounded continuous functions from a zero-dimensional topological space X to a non-Archimedean valued field $\mathbb{K}$. The dual space of $C_b(X)$ under the strict topology β_1 is investigated. Also certain subspaces $\theta_o X, \mu_o X$ of the Banachewski compactification $\beta_o X$ of X are introduced and some of the properties of the bounded finitely-additive measures on the algebra of all clopen subsets of X are studied.

Introduction

In [4], Alexandroff proved that the dual of the Banach space $C_b(X)$, of all bounded continuous real functions on a completely regular space X, can be identified with the space $M(X)$ of all bounded finitely-additive regular Baire measures on X. Several other authors have extended his results by considering various topologies on $C_b(X)$. Some of these topologies are called strict topologies and yield as dual spaces certain subspaces of $M(X)$. A good survey on Baire measures and strict topologies can be found in [19]. Some analogous problems have been studied in non-Archimedean spaces. For X a zero-dimensional Hausdorff topological space and $\mathbb{K}$ a complete non-Archimedean valued field, we denote by $C(X)$ the space of all continuous $\mathbb{K}$-valued functions on X and by $C_b(X)$ the subspace of all bounded members of $C(X)$. Let $M(X)$ be the space of all bounded finitely-additive measures on the algebra of all clopen subsets of X. An integration process of $\mathbb{K}$-valued functions on X with respect to members of $M(X)$ was defined in [8]. The strict topology β_o on $C_b(X)$ was defined in [9], the topologies β and β_1 were introduced in [10] and the topologies β_u and β_e in [1] and [12], respectively. As it was proved in the above papers, the topologies β_o, β, β_u and β_e yield as dual spaces certain subspaces of $M(X)$.
In section 2 of this paper, we prove that the dual space for the topology β_1 is the space

of all bounded σ-additive functionals on $C_b(X)$. It is also proved that a subset H of bounded linear functionals on $C_b(X)$ is β_1 (resp. β) equicontinuous iff it is norm bounded and uniformly σ-additive (resp. τ-additive). In section 3, if τ is a locally convex topology on $C_b(X)$, we will denote by τ^s the finest locally convex topology on $C_b(X)$ having the same as τ convergent sequences. It is proved that $\beta_1^s = \beta_1$. In case X is locally compact, it is shown that the topologies $\beta_o^s, \beta^s, \beta_1^s, \beta_e^s$ and β_u^s have the same dual space and that β_o^s coincides with the topology of uniform convergence iff X is pseudocompact. It is also proved that the topologies $\beta_o^s, \beta^s, \beta_1^s$ and β_u^s are locally solid.
In section 4, we introduce the subspaces $\theta_o X$ and $\mu_o X$ of $\beta_o X$. It is shown that $\theta_o X$ is the completion of X under the strongest non-Archimedean uniformity compatible with its topology and that $\mu_o X$ is the smallest of all subspaces Y of the **N**-repletion $\upsilon_o X$ of X which contain X and have the property that every bounded subset of Y is relatively compact. Finally in section 5 we study some of the properties of $M(X)$. If $M_\sigma(X), M_\tau(X)$ are the subspaces of all σ-additive and τ-additive members of $M(X)$, respectively, and $M_s(X)$ the space of separable measures, we give necessary and sufficient conditions on X so that one or more of the following equalities hold : $M_\sigma(X) = M_\tau(X), M_\sigma(X) = M_s(X), M(X) = M_\sigma(X), M(X) = M_s(X), M(X) = M_\tau(X)$.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is not trivial, X a Hausdorff zero-dimensional topological space, $C(X)$ the space of all continuous functions from X to $\mathbb{K}$, $C_b(X)$ the space of all bounded members of $C(X)$ and $C_{rc}(X)$ the space of all f in $C(X)$ whose range in $\mathbb{K}$ is relatively compact. For $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\|_A = \sup_{x \in A} |f(x)|, \|f\| = \|f|_X.$$

By a norm (resp. seminorm) on a vector space over $\mathbb{K}$ we mean a non-Archimedean norm (resp. seminorm). Also by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For $A \subset X, \chi_A$ is the $\mathbb{K}$-characteristic function of A. Let $K(X)$ denote the algebra of all clopen (i.e. both closed and open) subsets of X and let $M(X)$ (see [8]) be the space of all finitely-additive $\mathbb{K}$-valued measures on $K(X)$. For $m \in M(X)$ and A a clopen subset of X, we define

$$|m|(A) = \sup\{|m(B)| : B \in K(X), B \subset A\}, \|m\| = |m|(X).$$

For a net (V_δ) of subsets of X we write $V_\delta \downarrow \emptyset$ if it is decreasing and $\bigcap V_\delta = \emptyset$. We denote by $M_\sigma(X)$ the space of all σ-additive members of $M(X)$, i.e. those m such that $m(V_n) \to 0$ for each sequence (V_n) of clopen sets which decreases to the empty set (see [8]). The space $M_\tau(X)$ consists of all τ-additive members of $M(X)$, i.e. those m with $m(V_\delta) \to 0$ when $V_\delta \downarrow \emptyset$. If m is σ-additive and $A_n \downarrow \emptyset$, then $|m|(A_n) \to 0$ (see [8]). Also, if m is τ-additive and $V_\delta \downarrow \emptyset$, then $|m|(V_\delta) \to 0$ by [18], p. 249. For $m \in M(X), d$

a continuous ultrapseudometric on X on X and A a d-clopen subset of X, we define $|m|_d(A) = \sup |m(B)|$, where the supremum is taken over the family of all d-clopen subsets of A. Also, for Z a subset of X, we define $|m|_d^*(Z) = \inf \sup_n |m|_d(A_n)$, where the infimum is taken over the family of all sequences (A_n) of d-clopen subsets of X with $Z \subset \bigcup A_n$. We denote by $M_s(X)$ the space of all separable members of $M(X)$, i.e. those m with the property: for each continuous ultrapseudometric d on X, there exists a d-closed d-separable subset G of X with $|m|_d^*(G^c) = 0$ (see [12]). For $m \in M(X)$, we define

$$supp(m) = \bigcap\{V \in K(X) : |m|(V^c) = 0\}.$$

As it is shown in [8], if m is τ-additive, then $supp(m)$ is a support set for m, i.e. $m(V) = 0$ if the clopen set V is disjoint from $supp(m)$.
Next we will recall the definition of the integral of an $f \in \mathbb{K}^X$ with respect to a member m of $M(X)$ (see [7]). An $f \in \mathbb{K}^X$ is called m-integrable if there exists an element of $\mathbb{K}$, which we will denote by $\int f dm$ and call it the integral of f, such that for each $\epsilon > 0$ there exists a clopen partition $\{A_1, \ldots, A_n\}$ of X, such that for each clopen partition $\{B_1, \ldots B_N\}$ of X, which is a refinement of $\{A_1, \ldots, A_n\}$, and any choice of $x_i \in B_i$,m we have

$$\left| \int f dm - \sum_{i=1}^{N} f(x_i) m(B_i) \right| \leq \epsilon.$$

For $A \subset X$, we define $\int_A f dm = \int f \chi_A dm$. As it is shown in [12], if $m \in M_s(X)$, then every $f \in C_b(X)$ is m-integrable. In particular this happens if $m \in M_\tau(X)$.
Finally we will recall the definitions of the topologies $\beta_o, \beta, \beta_1, \beta_u$ on $C_b(X)$. Let $B_o(X)$ denote the space of all $\mathbb{K}$-valued functions on X which are bounded and vanish at infinity. Each $h \in B_o(X)$ defines a seminorm p_h on $C_b(X), p_h(f) = \|hf\|$. Then β_o is the locally convex topology generated by the seminorms $p_h, h \in B_o(X)$. Next, let Φ be a family of closed subsets of the Banachewski compactification $\beta_o X$ of X which are disjoint from X. Each $f \in C_{rc}(X)$ has a unique continuous extension f^{β_o} to all of $\beta_o X$. Let $Z \in \Phi$ and let C_Z be the set of all $f \in C_{rc}(X)$ with $f^{\beta_o} = 0$ on Z. We denote by β_Z the locally convex topology on $C_b(X)$ generated by the seminorms $p_h, h \in C_Z$. The inductive limit of the topologies $\beta_Z, Z \in \Phi$ will be denoted by β_Φ. Taking as Φ the collection Ω of all closed and the collection Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$ which are disjoint from X, we get the topologies β and β_1 respectively. The topology β_u is obtained by taking as Φ the collection Ω_u of all $Z \in \Omega$ for which there exists a clopen partition (V_i) on X such that, for each i, the closure $\overline{V_i}^{\beta_o X}$ of V_i in $\beta_o X$ is disjoint from Z (see [1]). As it is shown in [12], Theorem 2.2, an absolutely convex subset W of $C_b(X)$is a β_Z-neighborhood of zero, for some $Z \in \Omega$, iff for each $r > 0$ there exists a clopen subset A of X, whose closure in $\beta_o X$ is disjoint from Z, and an $\epsilon > 0$ such that $\{f : \|f\| \leq r, \|f\|_A \leq \epsilon\} \subset W$. Finally for the definition of the topology β_e we refer to [12].

2 σ-Additive and τ-Additive Functionals

Let G be a solid subspace of $C(X)$ and let u be a linear functional on G. Assume that u is bounded on each $B_f, f \in G$, where $B_f = \{g \in G : |g| \leq |f|\}$. Define $|u|$ on G by

$$|u|(f) = \sup\{|u(g)| : g \in B_f\}.$$

Then $|u|$ is a non-Archimedean seminorm on G. In fact it is clear that $|u|(\lambda f) = |\lambda||u|(f)$ for $f \in G, \lambda \in \mathbb{K}$. Let $f_1, f_2 \in G$ and $g \in G, |g| \leq \max\{|f_1|, |f_2|\}$. By [15], Proposition 2.4, there are $g_1, g_2 \in C(X), |g_i| \leq |f_i|, g = g_1 + g_2$. Since G is a solid subspace of $C(X)$, we have that $g_i \in B_{f_i}, i = 1, 2$. Hence

$$|u(g)| \leq \max\{|u(g_1)|, |u(g_2)|\} \leq \max\{|u|(f_1), |u|(f_2)\},$$

which proves that $|u|(f_1 + f_2) \leq \max\{|u|(f_1), |u|(f_2)\}$.

Theorem 2.1 *Let G be a solid subspace of $C(X)$ and let τ be a locally convex topology on G. Then, a linear functional u on G is τ-continuous iff it is bounded on each $B_f, f \in G$, and $|u|$ is τ-continuous.*

Proof: Assume that u is τ-continuous. Then, there exists a solid τ-neighborhhod W of zero with $W \subset \{f \in G : |u(f)| \leq 1\}$. Let $f \in G$. There exists a non-zero element $\lambda \in \mathbb{K}$ with $f \in \lambda W$. If $g \in G, |g| \leq |f|$, then $g \in \lambda W$ and so $|u(g)| \leq |\lambda|$. This proves that u is bounded on B_f. Moreover $|u|$ is τ-continuous since $W \subset \{f \in G : |u|(f) \leq 1\}$. Conversely, if u is bounded on each $B_f, f \in G$, and $|u|$ is τ-continuous, then the set $\{f : |u|(f) \leq 1\}$ is a τ-neighborhood of zero contained in $\{f : |u(f)| \leq 1\}$ and so u is τ-continuous.

For a net $f_\delta)$ in $\mathbb{K}^X$ we write $f_\delta \downarrow 0$ if, for each $x \in X$, the net $(|f_\delta(x)|)$ is decreasing to zero.A linear functional u on $C_b(X)$ is said to be τ-additive if for each net in $C_b(X)$ with $f_\delta \downarrow 0$ we have that $\lim u(f_\delta) \to 0$, and u is called bounded if it is continuous with respect to the topology τ_u of uniform convergence. For bounded u we define $\|u\| = \sup\{|u(f)|/\|f\| : f \in C_b(X), f \neq 0\}$.

Theorem 2.2 *For a collection H of bounded linear functionals on $C_b(X)$, the following are equivalent:*
(1) If $f_\delta \downarrow 0$, then $u(f_\delta) \to 0$ uniformly for $u \in H$.
(2) If $f_\delta \downarrow 0$, then $|u|(f_\delta) \to 0$ uniformly for $u \in H$.

Proof: Assume that (1) holds and that there exists a net $(f_\delta)_{\delta \in \Delta}$ in $\mathbb{K}^X$, with $f_\delta \downarrow 0$, and an $\epsilon > 0$ such that $\sup_{u \in H} |u|(f_\delta) > \epsilon$, for each δ.
Claim: For each δ there exists a $\beta \geq \delta, g \in C_b(X)$ and $u \in H$ such that $|f_\beta| \leq |g| \leq |f_\delta|$ and $|u(g)| > \epsilon$. In fact, there exists an $h, |h| \leq |f_\delta|$, and a $u \in H$ such that $|u(h)| > \epsilon$. For each $\beta \in \Delta$, define h_β on X by

$$h_\beta(x) = \begin{cases} f_\beta(x), & \text{if } |f_\beta(x)| \leq |h(x)| \\ h(x), & \text{if } |f_\beta(x)| > |h(x)| \end{cases}$$

As is is shown in the proof of Proposition 2.3 in [15], the function h_β is continuous. Let $g_\beta = f_\beta - h_\beta$. Since $|g_\beta| \leq |f_\beta|$ for all β, we have that $\lim g_\beta(x) = 0$ for all x. Also $|g_{\beta_2}| \leq |g_{\beta_1}|$ when $\beta_2 \geq \beta_1$. By our hypothesis, there exists $\beta \geq \delta$ such that $|u(g_\beta)| < |u(h)|$. Let now $g = h + g_\beta$. It is easy to see that $|f_\beta| \leq |g| = max\{|h|, |g_\beta|\} \leq |f_\delta|$. Moreover $|u(g)| = |u(h) + u(g_\beta)| = |u(h)| > \epsilon$, which proves our claim.
Let now $\mathcal{F}$ be the family of all $g \in C_b(X)$ with the following property: There are β, δ, with $\beta \geq \delta$, and $u \in H$ such that $|f_\beta| \leq |g| \leq |f_\delta|$ and $|u(g)| > \epsilon$. The family $\mathcal{F}$ is downwards directed. In fact, given $g_1, g_2 \in \mathcal{F}$, there are indices $\beta_i, \delta_i, i = 1, 2$ and $u_1, u_2 \in H$ such that $|f_{\beta_i}| \leq |g_i| \leq |f_{\delta_i}|$ and $|u_i(g_i)| > \epsilon$ for $i = 1, 2$. Let $\delta \geq \beta_1, \beta_2$. As we have shown above, there are $\beta \geq \delta, g \in C_b(X)$ and $u \in H$ such that $|f_\beta| \leq |g| \leq |f_\delta|$ and $|u(g)| > \epsilon$. Now $g \in \mathcal{F}$ and $|g| \leq |g_1|, |g_2|$. Also, given $\epsilon > 0$ and $x \in X$, there exists and index δ_o such that $|f_{\delta_o}(x)| < \epsilon$. By our claim, there exists $h \in \mathcal{F}$ with $|h(x)| \leq |f_{\delta_o}(x)|$. This proves that $\mathcal{F} \downarrow 0$ and so, by our hypothesis $\lim_{g \in \mathcal{F}} |u(g)| = 0$ uniformly for $u \in H$, which is a contradiction. This proves that (1) implies (2) and the result follows.

Theorem 2.3 *For a subset M of $M(X)$, the following are equivalent:*
(1) If $V_\delta \downarrow \emptyset$, then $m(V_\delta) \to o$ uniformly for $m \in M$.
(2) If $V_\delta \downarrow \emptyset$, then $|m|(V_\delta) \to o$ uniformly for $m \in M$.

Proof: Assume that (1) holds and let $(V_\delta)_{\delta \in \Delta}$ be a net of clopen subsets of X with $V_\delta \downarrow \emptyset$. Suppose that there exists $\epsilon > 0$ such that $\sup_{m \in M} |m|(V_\delta) > \epsilon$ for all δ.
Claim: For each $\delta \in \Delta$ there exist $\gamma \in \Delta, \gamma \geq \delta, m \in M$ and $D \in K(X)$ such that $V_\gamma \subset D \subset V_\delta$ and $|m(D)| > \epsilon$.
In fact, given δ, there exist $m \in H$ and a clopen subset V of V_δ such that $|m(V)| > \epsilon$. For each $\gamma \in \Delta$, let $Z_\gamma = V_\gamma \cap V, W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. By our hypothesis, there exists $\gamma \geq \delta$ such that $|m(W_\gamma)| < |m(V)|$. Let $D = V \cup W_\gamma$. Then $V_\gamma \subset D \subset V_\delta$. The sets V and W_γ are disjoint and so $|m(D)| = |m(V) + m(W_\gamma)| = |m(V)| > \epsilon$, which proves our claim.
Let now $\mathcal{F}$ be the family of all clopen subsets D of X with the following property: There are $\gamma, \delta \in \Delta$, with $\gamma \geq \delta$, and $m \in M$ such that $V_\gamma \subset D \subset V_\delta$ and $|m(D)| > \epsilon$. Using our claim we get that $\mathcal{F}$ is not empty and that $\mathcal{F} \downarrow \emptyset$. By our hypothesis, $\lim_{D \in \mathcal{F}} m(D) = 0$ uniformly for $m \in M$, which is a contradiction since, for each $D \in \mathcal{F}$, there exists $m \in M$ with $|m(D)| > \epsilon$. This clearly completes the proof.

In view of [12] Theorem 3.4, every $m \in M_\tau(X)$ defines a β-continuous linear funtional u_m on $C_b(X), u_m f) = \int d dm$, and that the map $m \to u_m$, from $M_\tau(X)$ to the dual space of $C_b(X)$ with the strict topology β, is an algebraic isomorphism. Moreover $\|u_m| = \|m\|$.

Theorem 2.4 *For a set H of bounded linear functionals on $C_b(X)$, the following are equivalent:*
(1) H is β-equicontinuous.
(2) There exists a subset M of $M_\tau(X)$ such that
  (a) $H = \{u_m : m \in H\}$.
  (b) $\sup_{m \in M} \|m\| < \infty$.

(c) *If $V_\delta \downarrow \emptyset$, then $m(V_\delta) \to 0$ uniformly for $m \in M$.*
(3) *There exists a subset M of $M_\tau(X)$ such that*
(a) $H = \{u_m : m \in H\}$.
(b) $\sup_{m\in M} \|m\| < \infty$.
(c) *If $V_\delta \downarrow \emptyset$, then $|m|(V_\delta) \to 0$ uniformly for $m \in M$.*
(4) *(a)* $\sup_{u\in H} \|u\| < \infty$.
(b) If $f_\delta \downarrow 0$, then $u(f_\delta) \to 0$ uniformly for $u \in H$.
(5) *(a)* $\sup_{u\in H} \|u\| < \infty$.
(b) If $f_\delta \downarrow 0$, then $|u|(f_\delta) \to 0$ uniformly for $u \in H$.

Proof: (1) is equivalent to (3) by [13], Theorem 2.6, (2) is equivalent to (3) by the preceding Theorem, and (4) is equivalent to (5) by Theorem 2.2.
(3) implies (4): In fact, let $f_\delta \downarrow 0, d > \sup_{m\in M} \|m\|, \epsilon > 0$. Without loss of generality we may assume that $\|f_\delta\| \leq 1$ for all δ. For each δ, let $V_\delta = \{x : |f_\delta(x)| \geq \epsilon/d\}$. Then $V_\delta \downarrow \emptyset$. Choose δ_o such that $|m|(V_\delta| < \epsilon$ for all $m \in M$ and all $\delta \geq \delta_o$. Now, for $\delta \geq \delta_o$ we have that $\left|\int f_\delta dm\right| \leq \epsilon$ and the implication follows.
(4) implies (1): Let W be the polar of H in $C_b(X)$. We will finish the proof by showing that W is a β_Z-neighborhood of zero for each $Z \in \Omega$. So let $Z \in \Omega$ and $r > 0$. There exists a decreasing net (V_δ) of clopen subsets of X with $\bigcap \overline{V_\delta}^{\beta_o X} = Z$. Then $\chi_{V_\delta} \downarrow 0$. Let $\epsilon > 0$ be such that $\epsilon\|u\| \leq 1$ for all $u \in H$. Choose $\mu \in \mathbb{K}, |\mu| \geq r$. By our hypothesis there exists δ such that $|u|(\chi_{V_\delta}) \leq 1/\mu|$ for all $u \in H$. The closure in $\beta_o X$ of the clopen set $V = X \setminus V_\delta$ is disjoint from Z. If $G = \{f \in C_b(X) : \|f\| \leq r, \|f\|_V \leq \epsilon\}$, then G is contained in W. Indeed, let $f \in G$. Then $f = g_1 + g_2, g_1 = f\chi_V, g_2 = f\chi_{V_\delta}$. Since , for $u \in H$,we have that $|u(g_1)| \leq \|u\|\|g_1\| \leq 1$, it follows that $g_1 \in W$. Also $|g_2| \leq |\mu\chi_{V_\delta}|$ and so $|u(g_2| \leq |\mu||u|(\chi_{V_\delta}) \leq 1$ which shows that $g_2 \in W$. Thus $f \in W$. This proves that W is a β_Z-neighborhood of zero and the result follows.

As a Corollary we get the following

Theorem 2.5 *The dual space of $(C_b(X), \beta)$ coincides with the space of all bounded τ-additive linear functionals on $C_b(X)$.*

Theorem 2.6 *For a subset H of bounded linear functionals on $C_b(X)$, the following are equivalent:*
(1) *H is uniformly σ-additive, i.e. if $f_n \downarrow 0$, then $u(f_n) \to 0$ uniformly for $u \in H$.*
(2) *If $f_n \downarrow 0$, then $|u|(f_n) \to 0$ uniformly for $u \in H$.*

Proof: Assume that (1) holds and that there exists a sequence (f_n) in $C_b(X)$ and $\epsilon > 0$ such that $\sup_{u\in H} |u|(f_n) > \epsilon$ for all n. As in the proof of Theorem 2.2, we show that for each n there exists $k > n, u \in H$ and $g \in C_b(X)$ such that $|f_k| \leq |g| \leq |f_n|, |u(g)| > \epsilon$. Let now $n_1 < n_2 < \ldots, m_k \in H, g_k \in C_b(X)$ be such that $|f_{n_{k+1}}| \leq |g_k| \leq |f_{n_k}|$ and $|m_k(g_k)| > \epsilon$. Since $g_k \downarrow 0$, we should have that $u(g_k) \to 0$ uniformly for $u \in H$, which is a contradiction. This clearly completes the proof.

The proof of the following Theorem is analogous to the proof of Theorem 2.3.

Theorem 2.7 *For a subset M of $M(X)$ the following are equivalent:*
(1) M *is uniformly σ-additive, i.e. if* $V_n \downarrow \emptyset$, *then* $m(V_n) \to 0$ *uniformly for* $m \in M$.
(2) *If* $V_n \downarrow \emptyset$, *then* $|m|(V_n) \to 0$ *uniformly for* $m \in M$.

Theorem 2.8 *For a subset H of bounded linear functionals on $C_b(X)$, the following are equivalent:*
(1) H *is β_1-equicontinuous.*
(2) *(a)* $\sup_{u \in H} \|u\| < \infty$.
(b) If $f_n \downarrow 0$, *then* $u(f_n) \to 0$ *uniformly for* $u \in H$.
(3) *(a)* $\sup_{u \in H} \|u\| < \infty$.
(b) If $f_n \downarrow 0$, *then* $|u|(f_n) \to 0$ *uniformly for* $u \in H$.

Proof: (2) is equivalent (3) in view of Theorem 2.6.
(1) $\Rightarrow$ (2): Assume that H is β_1-equicontinuous. Since β_1 is coarser than the topology τ_u of uniform convergence, it follows that H is τ_u-equicontinuous and so $\sup_{u \in H} \|u\| < \infty$. Let now W be the polar of H in $C_b(X)$ and let $f_n \downarrow 0$. By [12], Theorem 3.7, The sequence (f_n) is β_1-convergent to zero. If λ is a non-zero element of $\mathbb{K}$, then λW is a β_1-neighborhhood of zero. Thus there exists n_o such that $f_n \in \lambda W$ if $n \geq n_o$. Hence for $n \geq n_o$ we have $\sup_{u \in H} |u(f_n)| \leq |\lambda|$.
(3) $\Rightarrow$ (1) : For every $Z \in \Omega_1$ there exists a decreasing sequence (V_n) of clopen subsets of X such that $\bigcap \overline{V_n}^{\beta_o X} = Z$. Now the proof of the implication is analogous to the proof of the implication (5) $\Rightarrow$ (1) in Theorem 2.4.

As a Corollary we get the following

Theorem 2.9 *The dual space of $(C_b(X), \beta_1)$ coincides with the space of all bounded σ-additive linear functionals on $C_b(X)$.*

Notation 2.10 *a) For a net (f_δ) in $C_b(X)$ we write $f_\delta \downarrow^u 0$ if, for each $\epsilon > 0$, the net (V_δ^ϵ), $V_\delta^\epsilon = \{x : |f_\delta(x)| \geq \epsilon\}$ is decreasing and $\bigcap_\delta \overline{V_\delta^\epsilon}^{\beta_o X} \in \Omega_u$.*
b) For a decreasing net (V_δ), of clopen subsets of X, we write $V_\delta \downarrow^u \emptyset$ if $\bigcap \overline{V_\delta}^{\beta_o X} \in \Omega_u$.

Theorem 2.11 *If $f_\delta \downarrow^u 0$, then $f_\delta \xrightarrow{\beta_u} 0$.*

Proof: Without loss of generality we may assume that $\|f_\delta\| \leq 1$ for all δ. Let now W be a convex β_u-neighborhood of zero. Since β_u is coarser than τ_u, there exists an $\epsilon > 0$ such that $f \in W$ if $\|f\| \leq \epsilon$. Let $Z = \bigcap_\delta \overline{V_\delta}^{\beta_o X} \in \Omega_u$. There exist a clopen subset D of X, whose closure in $\beta_o X$ is disjoint from Z, and $\alpha > 0$ such that

$$\{f : \|f\| \leq 1, \|f\|_D \leq \alpha\} \subset W.$$

Since $\overline{D}^{\beta_o X}$ is disjoint from Z, there exists a δ such that $\overline{D}^{\beta_o X}$ is disjoint from $\overline{V_\delta}^{\beta_o X}$ and so $D \bigcap V_\delta^\epsilon = \emptyset$. Let now $\gamma \geq \delta$. Then $f_\gamma \chi_{V_\gamma}$ is zero on D and so $f_\gamma \chi_{V_\gamma} \in W$. Also $f_\gamma \chi_{V_\gamma}^c \in W$ since $\|f_\gamma \chi_{V_\gamma}^c\| \leq \epsilon$. Thus $f_\gamma \in W$ for all $\gamma \geq \delta$, which completes the proof.

Definition 2.12 *An element m of $M(X)$ is called u-additive if $m(V_\delta) \to$ zero if $V_\delta \downarrow^u \emptyset$.*

Lemma 2.13 *If m is u-additive, then $m \in M_s(X)$.*

Proof: Let $(A_i)_{i\in I}$ be a clopen partition of X. For each finite subset J of I, set $V_J = \bigcup_{i\in J} A_i, D_J = X \setminus V_J$. Then $D_J \downarrow^u \emptyset$ and so $m(X) - \sum_{i\in J} m(A_i) \to 0$. Thus $m \in M_s(X)$ by [12], Theorem 6.9.

The proofs of the following three Theorems are analogous to the ones of Theorems 2.2, 2.3, and 2.4 respectively.

Theorem 2.14 *for a subset H of bounded linear functionals on $C_b(X)$ the following are equivalent:*

(1) *If $V_\delta \downarrow^u \emptyset$, then $u(f_\delta) \to 0$ uniformly for $u \in H$.*
(2) *If $V_\delta \downarrow^u \emptyset$, then $|u|(f_\delta) \to 0$ uniformly for $u \in H$.*

Theorem 2.15 *For a subset M of $M(X)$ the following are equivalent:*

(1) *If $V_\delta \downarrow^u \emptyset$, then $m(V_\delta) \to 0$ uniformly for $m \in M$.*
(2) *If $V_\delta \downarrow^u \emptyset$, then $|m|(V_\delta) \to 0$ uniformly for $m \in M$.*

Theorem 2.16 *For a subset H of bounded linear functionals on $C_b(X)$, the following are equivalent:*

(1) *H is β_u-equicontinuous.*
(2) *H is β_e-equicontinuous.*
(3) *There exists a subset M of $M_s(X)$ such that*
 (a) $H = \{u_m : m \in M\}$.
 (b) $\sup\{\|m\| : m \in M\} < \infty$.
 (c) *If $V_\delta \downarrow^u \emptyset$, then $m(V_\delta) \to 0$ uniformly for $m \in M$.*
(4) *There exists a subset M of $M_s(X)$ such that*
 (a) $H = \{u_m : m \in M\}$.
 (b) $\sup\{\|m\| : m \in M\} < \infty$.
 (c) *If $V_\delta \downarrow^u \emptyset$, then $|m|(V_\delta) \to 0$ uniformly for $m \in M$.*
(5) (a) $\sup_{u\in H} \|u\| < \infty$.
 (b) *If $f_\delta \downarrow^u 0$, then $u(f_\delta) \to 0$ uniformly for $u \in H$.*
(6) (a) $\sup_{u\in H} \|u\| < \infty$.
 (b) *If $f_\delta^u \downarrow 0$, then $|u|(f_\delta) \to 0$ uniformly for $u \in H$.*

Let us say that a subset H of $G = (C_b(X), \tau_u)'$ is solid if, $u_o \in H$ and $u \in G$ with $|u| \le |u_o|$ imply that $u \in H$

Theorem 2.17 *If τ is any one of the topologies $\beta_o, \beta, \beta_1, \beta_u, \beta_e$, then the space $G\tau = (C_b(X), \tau)'$ is a solid norm-closed subspace of G.*

Proof: Let u be in the norm-closure of G_β and let $f_\delta \downarrow 0$. We need to show that $u(f_\delta) \to 0$. Without loss of generality we may assume that $\|f_\delta\| \leq 1$ for all δ. Given $\epsilon > 0$, choose $u_o \in G_\beta$ with $\|u - u_o\| \leq \epsilon$. There exists a δ_o such that $|u_o(f_\delta)| \leq \epsilon$ if $\delta \geq \delta_o$. Let now $\delta \geq \delta_o$. Then $|(u - u_o)(f_\delta)| \leq \|u - u_o\|\|f_\delta\| \leq \epsilon$ and so

$$|u(f_\delta)| \leq \max\{|(u - u_o(f_\delta)|, |u_o(f_\delta)| \leq \epsilon.$$

This proves that u is β-continuous by Theorem 2.5. Using Theorem 2.2, it follows that G_β is also solid. Hence the result follows for β. The proofs for the other topologies are analogous.

3 Sequential Convergence

For a locally convex topology τ on a vector space E over $\mathbb{K}$, we will denote by τ^s the finest locally convex topology on E having the same with τ convergent sequences. This is the locally convex topology which has as a base at zero the family of all absolutely convex subsets W of E with the following property: If a sequence (x_n) is τ-convergent to zero, then $x_n \in W$ eventually. Clearly τ^s is finer than τ and τ_1^s is finer than τ_2^s if τ_1 is finer than τ_2.

We have the following easily established

Lemma 3.1 *Let τ be a solid locally convex topology on $C_b(X)$ and let (f_n) be a τ-null sequence in $C_b(X)$. If $g_n \in B_{f_n}$ for all n, then (g_n) is τ-null.*

Theorem 3.2 *For a locally convex topology τ on $C_b(X)$, the following are equivalent:*
(1) τ^s is locally solid.
(2) If (f_n) is a τ-null sequence in $C_b(X)$ and if $g_n \in B_{f_n}$, then the sequence (g_n) is τ-null.

Proof: (1) $\Rightarrow$ (2): Let (f_n) be a τ-null sequence in $C_b(X)$ and let $g_n \in B_{f_n}$. Then (f_n) is τ^s-null and so (g_n) is τ^s-null by the preceding Lemma. Hence (g_n) is τ-null.

(2) $\Rightarrow$ (1): Let W be a convex τ^s-neighborhood of zero and let

$$V = \{f \in W : B_f \subset W\}.$$

Since W is balanced, it follows easily that V is balanced. Also, if $f_1, f_2 \in V$, then $f = f_1 + f_2 \in V$. Indeed, let $g \in B_f$. Since $|g| \leq \max\{|f_1|, |f_2|\}$, there exist (by [15], Proposition 2.4) $g_i \in B_{f_i}, i = 1, 2$, such that $g = g_1 + g_2$. Now $g_i \in W$ and so $g \in W$, which proves that $f \in V$ and hence V is absolutely convex. We claim that V is a τ^s-neighborhood of zero. Assume the contrary. Then there exists a τ-null sequence (f_n) with $f_n \notin V$ for all n. Since (f_n) is also τ^s-null, we may assume that $f_n \in W$ for all n. Now, for each n, there exists $g_n \in B_{f_n}$ with $g_n \notin W$. By our hypothesis, (g_n) is a τ-null sequence, which is a contradiction. This clearly completes the proof.

Corollary 3.3 *1) If τ is a locally solid topology on $C_b(X)$, then τ^s is locally solid.*
2) Each of the topologies $\beta_o^s, \beta^s, \beta_1^s, \beta_u^s$ is locally solid.

Theorem 3.4 $\beta_1^s = \beta_1$.

Proof: Let W be a β_1^s-neighborhood of zero. By the preceding Corollary, there exists a convex solid β_1^s-neighborhood V of zero contained in W. We will show that V is a β_Z-neighborhood of zero for each $Z \in \Omega_1$. So let $Z \in \Omega_1$. There exists a decreasing sequence (W_n) of clopen subsets of $\beta_o X$ with $\bigcap W_n = Z$. Let $V_n = W_n \cap X$ and λ be a non-zero element of $\mathbb{K}$. Then $\lambda\chi_{V_n} \downarrow 0$ and so the sequence $(\lambda\chi_{V_n})$ is β_1-convergent to 0 by [12] Theorem 3.7. Let n_o be such that $\lambda\chi_{V_n} \in V$ when $n \geq n_o$. Since $\beta_1^s \leq \tau_u^s = \tau_u$, there exists a non-zero element μ of $\mathbb{K}$ such that

$$G = \{g \in C_b(X) : \|g\| \leq |\mu|\} \subset V.$$

Let $n \geq n_o$. The closure in $\beta_o X$ of the clopen set $D = X \setminus V_n$ is disjoint from Z. Moreover, if

$$G_1 = \{f \in C_b(X) : \|f\| \leq |\lambda|, \|f\|_D \leq |\mu|\},$$

then $G_1 \subset V$. Indeed, let $f \in G_1$. Then $f = f_1 + f_2, f_1 = f\chi_{V_n}, f_2 = f\chi_D$. Now $f_1 \in V$ since $|f_1| \leq |\lambda\chi_{V_n}|$ and V is solid. Also $f_2 \in V$ since $\|f_2\| \leq |\mu|$. Thus $f \in V$. By [12], Theorem 3.2, V is a β_Z-neighborhood of zero. This, being true for each $Z \in \Omega_1$, implies that V is a β_1-neighborhood of zero, which completes the proof.

Theorem 3.5 *If X is locally compact, then the dual spaces of $C_b(X)$ under the topologies $\beta_o^s, \beta^s, \beta_u^s, \beta_e^s$ and β_1^s coincide.*

Proof: Clearly we only need to show that, if u is a bounded σ-additive linear functional on $C_b(X)$ and if a sequence (f_n) in $C_b(X)$ is β_o-convergent to the zero function, then $u(f_n) \to 0$. So let (f_n) be such a sequence. Then (f_n) is β_o-bounded and hence τ_u-bounded. But on τ_u-bounded sets the topology β_o coincides with the topology τ_c of uniform convergence on compact subsets of X, Thus (f_n) is τ_c-convergent to zero. Choose $\lambda \in \mathbb{K}, |\lambda| > \sup_n \|f_n\|$ and let $\epsilon > 0$. For each positive integer n, set

$$A_n = \bigcup_{k \geq n} \{x : |f_k(x)| \geq \epsilon/\|u\|\}.$$

Clearly A_n is open. Also it is closed. In fact, let $x_o \notin A_n$ and let Z be a compact neighborhood of x_o in X. There exists n_1 such that $\|f_k\|_Z < \epsilon/\|u\|$ if $k \geq n_1$. Choose a neighborhood Z_1 of x_o contained in Z such that $|f_k(x) - f_k(x_o)| < \epsilon/\|u\|$ for $k = 1, 2, \ldots, n_1, x \in Z_1$. Now $|f_k(x) - f_k x_o)| < \epsilon/\|u\|$ for all k and all $x \in Z_1$. Since $x_o \notin A_n$, there exists $k \geq n$ such that $|f_k(x_o)| < \epsilon/\|u\|$. If now $x \in Z_1$, then $|f_k(x)| = |f_k(x_o)| < \epsilon/\|u\|$ and so $x \notin A_n$. This proves that A_n is clopen. Moreover $A_n \downarrow \emptyset$. Since u is σ-additive, there exists n_o such that $|u|(\chi_{A_n}) < \epsilon/|\lambda|$ if $n \geq n_o$. Let now $n \geq n_o$. Then $|f_n\chi_{A_n}| \leq |\lambda\chi_{A_n}|$ and so $|u(f_n\chi_{A_n})| \leq |\lambda||u|(\chi_{A_n}) < \epsilon$. Also $\|f_n\chi_{A_n^c}\| \leq \epsilon/u$ and thus $|u(f_n\chi_{A_n^c})| \leq \epsilon$. Therefore, $|u(f_n)| \leq \epsilon$ for $n \geq n_o$ and the result follows.

Theorem 3.6 *If X is locally compact, then $\beta_o^s = \tau_u$ iff X is $\mathbb{K}$-pseudocompact (equivalently pseudocompact).*

Proof: Assume that $\beta_o^s \neq \tau_u$. Then the set $W = \{f \in C_b(X) : \|f\| \leq 1\}$ is not a β_o^s-neighborhood of zero. There exists a β_o-null sequence (f_n) with $f_n \notin W$ for all n. Let $n_1 = 1$. Choose x_1 such that $|f_{n_1}(x_1)| > 1$. There exists $n_2 > n_1$ such that $|f_k(x_1)| < 1$ if $n \geq n_2$. Inductively we choose $n_1 < n_2 <, \dots,$ and a sequence (x_k) such that $max_{1\leq j\leq k}|f_n(x_j| < 1$ if $n \geq n_{k+1}$ and $|f_{n_{k+1}}(x_{k+1})| > 1$. Note that (f_n) is τ_u-bounded and so it converges to zero in the topology τ_c. Let $g_k = f_{n_k}$ and $A_k = \bigcup_{j\geq k}\{x : |g_j(x)| \geq 1\}$. As in the proof of the preceding Theorem, A_k is clopen. Let $D_k = A_k \setminus A_{k+1}$. Then $x_k \in D_k$ and $\{A_1^c, D_1, D_2, \dots\}$ is an infinite clopen partition of X and so X is not $\mathbb{K}$-pseudocompact. Conversely, assume that $\beta_o^s = \tau_u$ and that X is not $\mathbb{K}$-pseudocompact. Then there exists an infinite countable clopen partition (Z_n) of X. If $g_n = \chi_{Z_n}, f_n = g_1+g_2+\dots,+g_n$, then $f_n \to f$ in the topology τ_c, where $f(x) = 1$ for all x. But then $f_n \to f$ in the topology β_o and so (by our hypothesis) $f_n \to f$ in the topology τ_u. Thus, there exists n_o such that $\|f - f_n\| < 1$ if $n \geq n_o$. But if $n \geq n_o$ and $x \in Z_n$, then $1 = |f_{n_o}(x) - 1| < 1$, a contradiction. Hence the result follows.

Corollary 3.7 *If X is locally compact and pseudocompact, then every bounded linear functional on $C_b(X)$ is σ-additive.*

4 The Spaces $\theta_o X$ and $\mu_o X$

Let Φ_o be the collection of all finite clopen partitions of X and let Φ be any collection of clopen partitions of X which contains Φ_o and which is directed in the sense that, if α_1, α_2 are in Φ, then there exists $\alpha \in \Phi$ which is a refinement of both α_1 and α_2. For $\alpha = (V_i)_{i\in I}$ in Φ, we set

$$W_\alpha = \bigcup\{V_i \times V_i : i \in I\}.$$

Then the family $\{W_\alpha : \alpha \in \Phi\}$ is a base for a uniformity $\mathcal{U}_\Phi$ on X which is compatible with the topology of X. We will refer to $\mathcal{U}_\Phi$ as the uniformity generated by Φ. Let Φ_c be the collection of all clopen partitions of X and let Φ_1 be the subcollection of all countable members of Φ_c. We will denote by $\mathcal{U}_o = \mathcal{U}_o^X, \mathcal{U}_1 = \mathcal{U}_1^X$ and $\mathcal{U}_c = \mathcal{U}_c^X$ the uniformities generated by Φ_o, Φ_1, Φ_c, respectively. It is well known (see [18]) that $(\upsilon_o X, \mathcal{U}_1^{\upsilon_o X})$ and $(\beta_o X, \mathcal{U}_o^{\beta_o X})$ are the completions of $(X, \mathcal{U}_1)$ and $(X, \mathcal{U}_o)$ respectively. Since $\beta_o X$ is compact, there is only one compatible uniformity on $\beta_o X$. We will look at the completion of $(X, \mathcal{U}_c)$.

Notation 4.1 *We will denote by $\theta_o X$ the set of all $x \in \beta_o X$ with the following property: For each clopen partition $(V_i)_{i\in I}$ of X, there exists an i such that $x \in \overline{V_i}^{\beta_o X}$. Equivalently*

$$\theta_o X = \bigcap_{H\in\Omega_u} (\beta_o X \setminus H).$$

Lemma 4.2 $X \subset \theta_o X \subset \upsilon_o X$.

Proof: It is clear that X is contained in $\theta_o X$. Let $x \in \beta_o X \setminus \upsilon_o X$. There exists a decreasing sequence (V_n) of clopen neighborhods of x in $\beta_o X$ with $\bigcap_n V_n \cap X = \emptyset$. Let $W_n = V_n \cap X, D_n = X \setminus W_n, Z_1 = D_1$ and $Z_{n+1} = D_{n+1} \setminus D_n$ for $n \geq 1$. Then (Z_n) is a clopen partition of X. Since $Z_n \cap W_n = \emptyset$, it follows that $\emptyset = \overline{W_n}^{\beta_o X} \cap \overline{Z_n}^{\beta_o X} = V_n \cap \overline{Z_n}^{\beta_o X}$ and so $x \notin \overline{Z_n}^{\beta_o X}$, which implies that $x \notin \theta_o X$. Hence the result follows.

If now (A_i) is a clopen partition of X, then the family $(\overline{A_i}^{\theta_o X})$ is a clopen partition of $\theta_o X$. Conversely, if (B_i) is a clopen partition of $\theta_o X$ and if $D_i = B_i \cap X$, then (D_i) is a clopen partition of X and $B_i = \overline{D_i}^{\theta_o X}$. Thus $\mathcal{U}_c$ coincides with the uniformity induced on X by $\mathcal{U}_c^{\theta_o X}$.

Theorem 4.3 $(\theta_o X, \mathcal{U}_c^{\theta_o})$ *is the completion of* $(X, \mathcal{U}_c)$.

Proof: Since $\mathcal{U}_c^{\theta_o X}$ is compatible with the topology of $\theta_o X$ and X is dense in $\theta_o X$, it only remains to show that $\mathcal{U}_c^{\theta_o X}$ is complete. So let (y_δ) be a net in $\theta_o X$ which is $\mathcal{U}_c^{\theta_o X}$-Cauchy. Then (y_δ) is $\mathcal{U}_o^{\beta_o X}$-Cauchy and hence there exists $y \in \beta_o X$ such that $y_\delta \to y$ in $\beta_o X$. We only need to show that $y \in \theta_o X$. So, let (A_i) be a clopen partition of X and $B_i = \overline{A_i}^{\theta_o X}$. Then (B_i) is a clopen partition of $\theta_o X$. Since $G = \bigcup B_i \times B_i$ is in $\mathcal{U}_c^{\theta_o X}$ and (y_δ) is $\mathcal{U}_c^{\theta_o X}$-Cauchy, there exists δ_o such that $(y_\delta, y_{\delta_o}) \in G$ if $\delta \geq \delta_o$. Let $\imath_o$ be such that $y_{\delta_o} \in B_{\imath_o}$. Then $y_\delta \in B_{i_o}$ if $\delta \geq \delta_o$. Thus $y \in \overline{B_{i_o}}^{\beta_o X} = \overline{A_{i_o}}^{\beta_o X}$. This proves that $y \in \theta_o X$ and the result follows.

Definition 4.4 *We will say that* X *is* θ_o*-complete if* $X = \theta_o X$, *equivalently if the uniform space* $(X, \mathcal{U}_c)$ *is complete.*

Theorem 4.5 *Let* X, Y *be Hausdorff zero-dimensional topological spaces and let* $f : X \longrightarrow Y$ *be a continuous function. Then for the continuous extension* $f^{\beta_o} : \beta_o X \longrightarrow \beta_o Y$, *we have that* $f^{\beta_o}(\theta_o X) \subset \theta_o X$ *and so there exists a continuous extension* $f^{\theta_o} : \theta_o X \longrightarrow \theta_o Y$ *of* f.

Proof: Let $x \in \theta_o X$ and let (B_i) be a clopen partition of Y. If $A_i = f^{-1}(B_i)$, then (A_i) is a clopen partition of X. There exists an i such that $x \in \overline{A_i}^{\beta_o X}$. Thus

$$f_{\beta_o}(x) \in f_{\beta_o}(\overline{A_i}^{\beta_o X}) \subset \overline{f(A_i}^{\beta_o Y} \subset \overline{B_i}^{\beta_o Y},$$

which proves that $f_{\beta_o}(x) \in \theta_o Y$. Hence the result follows.

Theorem 4.6 *If* X *is ultraparacompact, then* $\theta_o X = X$.

Proof: Let $x \in \beta_o X \setminus X$ and $H = \{x\}$. Since X is ultraparacompact, we have that $\Omega = \Omega_u$ and so $H \in \Omega_u$, which implies that $x \notin \theta_o X$.

Recall that a subset A of X is called $\mathbb{K}$-bounded, or simply bounded, if every $f \in C(X)$ is bounded on A. In view of [11], Proposition 3.1, A is bounded iff the closure of A in

$\upsilon_o X$ is compact. Hence the notion of $\mathbb{K}$-boundedness does not depend on the particular choice of $\mathbb{K}$.

Theorem 4.7 *Let A be a bounded subset of X. Then:*

(i) A is $\mathcal{U}_c$-precompact and hence $\mathcal{U}_1$-precompact.

(ii) $\overline{A}^{\theta_o X} = \overline{A}^{\upsilon_o X} = \overline{A}^{\beta_o X}$.

(iii) $\mathcal{U}_c = \mathcal{U}_1$ on A.

Proof: (i) Assume that A is not $\mathcal{U}_c$-precompact. Then, there exists a clopen partition $\alpha = (A_i)$ of X such that, for $W = W_\alpha$, A is not contained in $W[S]$ for any finite subset S of A. If $x \in A_i$, then $W[x] = A_i$. Hence there exists a sequence (x_n)in A such that $W[x_n] \cap W[x_m] = \emptyset$ if $n \neq m$. For each n there exists an i_n such that $W[x_n] = A_{i_n}$. Let $\lambda \in \mathbb{K}, |\lambda| > 1$ and $f = \sum_{n=1}^{\infty} \lambda^n \chi_{A_{i_n}}$. Then f is continuous but not bounded on A, which is a contradiction.

(ii), (iii) The set $B = \overline{A}^{\theta_o X}$ is $\mathcal{U}_c^{\theta_o}$-precompact since A is $\mathcal{U}_c$-precompact. Also B is $\mathcal{U}_c^{\theta_o}$-complete. Thus B is compact in $\theta_o X$, which implies that B is compact in $\upsilon_o X$. Therefore $\overline{A}^{\theta_o X} = \overline{A}^{\upsilon_o X} = \overline{A}^{\beta_o X}$. Moreover $\mathcal{U}_c^{\theta_o X} = \mathcal{U}_1^{\theta_o X}$ on B (since B is compact) and hence $\mathcal{U}_c = \mathcal{U}_1$ on A. This completes the proof.

Lemma 4.8 *If B is a bounded subset of $\theta_o X$, then $A = B \cap X$ is a bounded subset of X.*

Proof: If $Y = \theta_o X$, then $\upsilon_o X = \upsilon_o Y$. Now

$$\overline{A}^{\upsilon_o Y} = \overline{A}^{\upsilon_o X} \subset \overline{B}^{\upsilon_o X} = \overline{B}^{\upsilon_o Y}.$$

Since $\overline{B}^{\upsilon_o Y}$ is compact, its closed subset $\overline{A}^{\upsilon_o X}$ is compact and so A is a bounded subset of X.

Definition 4.9 *We say that X is a μ_o-space if every bounded subset of X is relatively compact.*

Theorem 4.10 $\theta_o(\theta_o X) = \theta_o X$.

Proof: Let $Y = \theta_o X$. Then $\upsilon_o X = \upsilon_o Y$ and $\beta_o X = \beta_o Y$. Assume that there exists an $x \in \theta_o Y \setminus Y$. Since $x \notin Y$, there exists a clopen partition (A_i) of X such that $x \notin \overline{A_i}^{\beta_o X}$ for all i. If $B_i = \overline{A_i}^{\theta_o X} = \overline{A_i}^{\beta_o X} \cap Y$, then (B_i) is a clopen partition of Y. Since $x \in \theta_o Y$, there exists an i such that $x \in \overline{B_i}^{\beta_o X}$, which is a contradiction since $\overline{B_i}^{\beta_o X} \subset \overline{A_i}^{\beta_o X}$. Hence the result follows.

Theorem 4.11 *Both $\theta_o X$ and $\upsilon_o X$ are μ_o-spaces.*

Proof: Let $Y = \theta_o X, Z = \upsilon_o X$. Then $\upsilon_o Y = \upsilon_o X, \upsilon_o Z = Z, \beta_o Z = \beta_o X$. If B is a bounded subset of Y, then $\overline{B}^Y = \overline{B}^{\theta_o Y}$ and so $\overline{B}^Y$ is compact, which proves that Y is a μ_o-space. Similarly, let A be a bounded subset of Z. Then $\overline{A}^Z = \overline{A}^{\upsilon_o Z}$ and so $\overline{A}^Z$ is compact. Thus the result follows.

Let

$$\mu_o X = \bigcap \{Y : X \subset Y \subset \upsilon_o X, Y \text{a } \mu_o\text{-space}\}.$$

For $G = \mu_o X$, we have that $\upsilon_o G = \upsilon_o X$. Let A be a bounded subset of G. Then $\overline{A}^{\upsilon_o G} = \overline{A}^{\upsilon_o X}$ is compact. If Y is a μ_o-space with $X \subset Y \subset \upsilon_o X$, then A is bounded in Y and so $\overline{A}^{Y}$ is compact. Thus $\overline{A}^{\upsilon_o G} = \overline{A}^{Y}$ and so

$$\overline{A}^{\upsilon_o G} = \bigcap \{\overline{A}^{Y}, X \subset Y \subset \upsilon_o X, Y \text{ a } \mu_o\text{-space}\} \subset G$$

and so $\overline{A}^{G} = \overline{A}^{\upsilon_o G}$ is compact, which proves that G is a μ_o-space. Clearly $\mu_o X$ is the smallest of all μ_o-subspaces of $\upsilon_o X$ which contain X. Moreover $\mu_o X \subset \theta_o X$. Thus X is a μ_o-space iff $X = \mu_o X$.

Theorem 4.12 *Let X, Y be Hausdorff zero-dimensional topological spaces and let $f : X \longrightarrow Y$ be a continuous function. Then for the continuous extension $f^{\beta_o} : \beta_o X \longrightarrow \beta_o Y$, we have that $f^{\beta_o}(\mu_o X) \subset \mu_o X$ and so there exists a continuous extension $f^{\mu_o} : \mu_o X \longrightarrow \mu_o Y$ of f.*

Proof: Let $Z = (f^{\upsilon_o})^{-1}(\mu_o Y)$. Then $X \subset Z \subset \upsilon_o X$. Moreover Z is a μ_o-space. Indeed, let A be a bounded subset of Z. Then, A is bounded in $\upsilon_o Z = \upsilon_o X$ and so $\overline{A}^{\upsilon_o X} = \overline{A}^{\upsilon_o Z}$ is compact. The set $B = f^{\upsilon_o}(A)$ is bounded in $\mu_o Y$ and so $\overline{B}^{\mu_o Y}$ is compact. Also,

$$f^{\upsilon_o}(\overline{A}^{\upsilon_o X}) \subset \overline{B}^{\upsilon_o Y} = \overline{B}^{\mu_o Y} \subset \mu_o Y$$

and hence $\overline{A}^{\upsilon_o X} \subset Z$, which implies that $\overline{A}^{Z} = \overline{A}^{\upsilon_o X}$ is compact. Thus Z is a μ_o-space and so $\mu_o X \subset Z$. This completes the proof.

Theorem 4.13 *Closed subspaces and Cartesian products of μ_o-spaces are μ_o-spaces.*

Proof: It is easy to see that a closed subspace of a μ_o-space is a μ_o-space.
Let (X_i) be a family of Hausdorff zero-dimensional μ_o-spaces and let A be a bounded subset of $Z = \prod X_i$. If $\pi_i : Z \to X_i$ is the *ith*-projection, then $A_i = \pi_i(A)$ is a bounded subset of X_i. Thus $\overline{A_i}$ is compact and so $B = \prod B_i$ is compact. Since $A \subset B$, the result follows.

5 On the Space $M(X)$

Theorem 5.1 *The following are equivalent :*

(1) $M_\sigma(X) = M_\tau(X)$.

(2) *X is N-replete.*

(3) *For every non-zero $m \in M_\sigma(X)$ we have that $supp(m) \neq \emptyset$.*

(4) *For every non-zero $m \in M_\sigma(X)$, $supp(m)$ is a support set for m, i.e. $m(V) = 0$ if the clopen set V is disjoint from $supp(m)$.*

Proof: The equivalence of (1) and (2) is proved in [18], pp 250-251. Also (2) $\Longrightarrow$ (4), by [7], Theorem 3.5, and the implication (4) $\Longrightarrow$ (3) is trivial.
(3) $\Longrightarrow$ (2): Suppose that there exists an $x \in \upsilon_o X \setminus X$. Let $\mu \in M(\beta_o X)$ be defined by $\mu(W) = 1$ if $x \in W$ and $\mu(W) = 0$ otherwise. Let

$$m : K(X) \longrightarrow \mathbb{K}, m(V) = \mu(\overline{V}^{\beta_o X}).$$

Then $m \in M_\sigma(X)$. Indeed, let (V_n) be a sequence of clopen subsets of X which decreases to the empty set. Then $x \notin \bigcap(\overline{V_n}^{\beta_o X}$ since $x \in \upsilon_o X$. It follows from this that $m(V_n) = 0$ finally and so m is σ-additive. By our hypothesis there exists a $y \in supp(m)$. Since $y \neq x$, there are disjoint clopen neighborhoods D_1, D_2 of y, x in $\beta_o X$. If V is a clopen subset of X contained in $W = D_1 \cap X$, then $x \notin \overline{V}^{\beta_o X}$ and so $m(V) = 0$, which implies that $|m|(W) = 0$. But then $supp(m) \subset X \setminus W$, which leads to a contradiction since $y \in W$. Hence the result follows.

We will denote by $M_b(X)$ the space of all $m \in M(X)$ for which there exists a bounded subset B of X which is a support set for m, i.e. $m(V) = 0$ if V is disjoint from B.

Theorem 5.2 $M_b(X) \subset M_s(X)$.

Proof: Let $m \in M_b(X)$ and let A be a bounded subset of X which is a a support set for m. Then $\overline{A}^{\beta_o X} = \overline{A}^{\theta_o X} \subset \theta_o X$. If $(V_i)_{i \in I}$ is a clopen partition of X, then $(\overline{V_i}^{\theta_o X})$ is a clopen partition of $\theta_o X$. Since $\overline{A}^{\beta_o X}$ is compact, there exists a finite subset J of I such that $\overline{A}^{\beta_o X} \subset \bigcup_{i \in J} \overline{V_i}^{\theta_o X}$ and so $A \subset \bigcup_{i \in J} V_i = D$. If $B = \bigcup_{i \notin J} V_i$, then $m(B) = 0$ and $m(V_i) = 0$ if $i \notin J$. Hence

$$m(X) = m(D) + m(B) = \sum_{i \in J} m(V_i) = \sum_{i \in I} m(V_i).$$

Thus $m \in M_s(X)$ in view of [12], Theorem 6.9, and we are done.

Let $m \in M(X)$. For Y a subspace of $\beta_o X$ containing X, we define $m^Y \in M(Y)$ by $m^Y(V) = m(V \cap X)$. In case Y is one of the spaces $\theta_o X, \mu_o X, \upsilon_o X, \beta_o X$, we write $m^{\theta_o}, m^{\mu_o}, m^{\upsilon_o}, m^{\beta_o}$ for m^Y, respectively.

Theorem 5.3 *a) The map $m \mapsto m^{\theta_o}$, from $M_s(X)$ to $M_s(\theta_o X)$, is an algebraic isomorphism.*

b) The map $m \mapsto m^{\upsilon_o}$, from $M_\sigma(X)$ to $M_\sigma(\upsilon_o X)$, is an algebraic isomorphism.

c) The map $m \mapsto m^{\beta_o}$, from $M(X)$ to $M(\beta_o X)$, is an algebraic isomorphism.

Proof; a) It follows from [12], Theorem 6.9, since, for each clopen partition (V_i) of X, the family $(\overline{V_i}^{\theta_o X})$ is a clopen partition of $\theta_o X$.
b) If W is a clopen subset of $\upsilon_o X$ and $V = W \cap X$, then $\overline{V}^{\upsilon_o X} = W$. Now (*b*) follows from the fact that, if (V_n) is a sequence of clopen subsets of X which decreases to the empty set, then $(\overline{V_n}^{\upsilon_o X})$ is a sequence of clopen subsets of $\upsilon_o X$ which decreases to the

empty set.
c) It is trivial.

Theorem 5.4 *$M_\sigma(X) = M_s(X)$ iff $\theta_o X = \upsilon_o X$.*

Proof: Assume that there exists an $x \in \upsilon_o X$ which is not in $\theta_o X$. Define m on $K(X)$ by $m(V) = 1$ if $x \in \overline{V}^{\upsilon_o X}$ and $m(V) = 0$ otherwise. If (V_n) is a sequence of clopen subsets of X which decreases to the empty set, then $x \notin \bigcap \overline{V_n}^{\upsilon_o X}$, which implies that $m(V_n) = 0$ eventually. This proves that m is σ-additive. Since $x \notin \theta_o X$, there exists a clopen partition (A_i) of X such that $x \notin \overline{A_i}^{\beta_o X}$ for all i. Now $m(X) = 1$ and $m(A_i) = 0$, which implies that $m \notin M_s(X)$. Conversely, assume that $\theta_o X = \upsilon_o X$ and let $m \in M_\sigma(X)$. Then $m^{\upsilon_o} \in M_\sigma(\upsilon_o X)$. As $\upsilon_o X$ is **N**-replete, we have that $m^{\upsilon_o} \in M_\tau(\upsilon_o X) \subset M_s(\upsilon_o X) = M_s(\theta_o X)$ and so $m \in M_s(X)$. This completes the proof.

Theorem 5.5 *The following are equivalent :*

(1) $M(X) = M_\sigma(X)$.
(2) $M(X) = M_s(X)$.
(3) $\upsilon_o X = \beta_o X$.
(4) $\theta_o X = \beta_o X$.
(5) *X is $\mathbb{K}$-pseudocompact (equivalently pseudocompact).*
(6) *Every clopen partition of X is finite.*
(7) *Every countable clopen partition of X is finite.*

Proof: It is easy to see that (6) is equivalent to (7). Also (1), (3) and (5) are equivalent by [12], Theorem 2.9, and [11], Proposition 3.3. In view of [5], Theorem 1.1, (5) is equivalent to (6).
(6) $\Rightarrow$ (4) If every clopen partition of X is finite, then $\mathcal{U}_c = \mathcal{U}_o$ and so the completions of $(X, \mathcal{U}_c)$ and $(X, \mathcal{U}_o)$ coincide, i.e. $\theta_o X = \beta_o X$.
(4) $\Rightarrow$ (6) Assume that $\theta_o X = \beta_o X$ and let $(A_i)_{i \in I}$ be a clopen partition of X . Then the family $(\overline{A_i}^{\beta_o X})_{i \in I}$ be a clopen partition of the compact space $\theta_o X = \beta_o X$. There exists a finite subset J of I such that $\beta_o X = \bigcup_{i \in J} \overline{A_i}^{\beta_o X}$. If now $i \notin J$, then $A_i = \emptyset$.
(6) $\Rightarrow$ (2) Let $m \in M(X)$ and let $(A_i)_{i \in I}$ be a clopen partition of X. Then I is finite and so $m(X) = \sum m(A_i)$, which proves that $m \in M_s(X)$ by [12], Theorem 6.9.
Since (2) trivially implies (1), the result follows.

Theorem 5.6 *$M(X) = M_\tau(X)$ iff X is compact.*

Proof: It is clear that the condition is sufficient. To prove the necessity, suppose that $M(X) = M_\tau(X)$. Then $M(X) = M_\sigma(X) = M_\tau(X)$ and so $\upsilon_o X = \beta_o X$ and $X = \upsilon_o X$ by Theorems 5.5 and 5.1, and so $X = \beta_o X$, i.e. X is compact.

Theorem 5.7 *(1) If $M_\tau(X) = M_s(X)$, then $\theta_o X = X$.*
(2) If each clopen partition of X has non-measurable cardinal (in particular if X has non-measurable cardinal), then the following are equivalent:

(a) $\theta_o X = X$.
(b) $M_\tau(X) = M_s(X)$.
(c) X *is* **N**-*replete.*
(d) $M_\tau(X) = M_\sigma(X)$.

Proof: (1) Assume that there exists an $x \in \theta_o X \setminus X$. Define m on $K(X)$ by $m(V) = 1$ if $x \in \overline{V}^{\beta_o X}$ and $m(V) = 0$ otherwise. If $(A_i)_{i\in I}$ is a clopen partition of X, then the sets $\overline{A_i}^{\beta_o X}, i \in I$ are pairwise disjoint and there is a unique i_o such that $x \in \overline{A_{i_o}}^{\beta_o X}$. Thus $\sum_{i\in I} m(A_i) = m(A_{i_o}) = 1 = m(X)$, which proves that $m \in M_s(X)$. Now there exists a decreasing net (W_δ) of clopen subsets of $\beta_o X$ with $\bigcap_\delta W_\delta = \{x\}$. If $V_\delta = W_\delta \cap X$, then $V_\delta \downarrow \emptyset$ and $m(V_\delta) = 1$ for all δ. Hence m is not τ-additive.
(2) We know that (c) is equivalent to (d). Also (c) implies (a) since $X \subset \theta_o X \subset \upsilon_o X$. If $\theta_o X = X$, then the uniformity $\mathcal{U}$ is complete and hence X is **N**-replete by [18], Theorem 2.11. Thus (a) implies (c). It is clear that (d) implies (b). Also (b) implies (a) by (1). This completes the proof.

Theorem 5.8 *If $m \in M(X)$, then $m \in M_b(X)$ iff there exists a bounded subset B of X such that $supp(m^{\beta_o}) \subset \overline{B}^{\beta_o X} = \overline{B}^{\upsilon_o X}$.*

Proof: Assume that $m \in M_b(X)$ and let B be a bounded support set for m. Let $x \in G = \beta_o X \setminus \overline{B}^{\beta_o X}$ and let W be a clopen neighborhood of x in $\beta_o X$ contained in G. If $V \in K(\beta_o X)$ is contained in W and $D = V \cap X$, then D is disjoint from B and so $0 = m(D) = m^{\beta_o}(V)$. This proves that $|m^{\beta_o}|(W) = 0$ and so $supp(m^{\beta_o}) \subset \beta_o X \setminus W$, which implies that x is not in $supp(m^{\beta_o})$. Hence $supp(m^{\beta_o}) \subset \overline{B}^{\beta_o X}$.
Conversely, assume that $supp(m^{\beta_o}) \subset \overline{B}^{\beta_o X}$ for some bounded subset B of X. Since the closure Z of B in X is also bounded and $\overline{B}^{\beta_o X} = \overline{Z}^{\beta_o X}$, we may assume that B is closed in X. Let now V be a clopen subset of X disjoint from B. If $W = \overline{V}^{\beta_o X}$, Then W is disjoint from $\overline{B}^{\beta_o X}$. In fact, suppose that some point x of W is contained in $\overline{B}^{\beta_o X}$. There exists a net (x_δ) in B converging to x. Since W is open in $\beta_o X$, there exists a δ_o such that $x_\delta \in W$ if $\delta \geq \delta_o$. But then, for $\delta \geq \delta_o$ we have that $x_\delta \in W \cap B = (W \cap X) \cap B = V \cap B = \emptyset$, a contradiction.Thus W is disjoint from $\overline{B}^{\beta_o X}$ and so W is disjoint from $supp(m^{\beta_o})$. Hence $m(V) = m(W) = 0$, which proves that B is a support set for m. Thus $m \in M_b(X)$ and the result follows.

Let $\mathcal{F}$ be the family of all bounded partitions of unity on X and let $\mathcal{F}_{rc}$ be the subfamily of all members of $\mathcal{F}$ contained in $C_{rc}(X)$. Let $m \in M_s(X)$ and let u_m be the corresponding bounded linear functional on $C_b(X)$. For $\omega = (f_i)_{i\in I} \in \mathcal{F}$, let

$$J_m(\omega) = \{i \in I : |u_m|(f_i) \neq 0\}, \ S_m(\omega) = \bigcup_{i\in J_m(\omega)} \{x : f_i(x) \neq 0\}.$$

Theorem 5.9 *For $m \in M_s(X)$, we have*

$$supp(m) = \bigcap_{\omega\in\mathcal{F}} S_m(\omega) = \bigcap_{\omega\in\mathcal{F}_{rc}} S_m(\omega).$$

Proof: Suppose that $x \notin supp(m)$. There exists a clopen neighborhood V of x with $|m|(V) = 0$. Take $\omega = \{\chi_V, \chi_{V^c}\}$. Since $|u_m|(\chi_V) = |m|(V) = 0$, we have that $S_m(\omega) \subset \{x : \chi_{V^c}(x) \neq 0\} = V^c$ and so $x \notin S_m(\omega)$. Conversely, suppose that $x \in supp(m)$ and let $\omega = (f_i)_{i \in I} \in \mathcal{F}$. There exists i such that $f_i(x) \neq 0$. The set

$$W = \{y : |f_i(y)| \geq |f_i(x)|\}$$

is a clopen neighborhood of x. As $x \in supp(m)$, we have that $|m|(W) \neq 0$. Now for $\gamma = f_i(x)$, we have that $|\gamma \chi_W| \leq |f_i|$ and so

$$|u_m|(f_i) \geq |u_m|(\gamma \chi_W) = |\gamma||u_m|(\chi_W) = |\gamma||m|(W).$$

Thus $i \in J_m(\omega)$ and $f_i(x) \neq 0$, which implies that $x \in S_m(\omega)$. This clearly completes the proof.

Notation 5.10 *For $\mu \in M(\beta_o X)$ and $D \subset \beta_o X$ we define*

$$|\mu|_*(D) = \sup \inf_n |\mu|(W_n),$$

where the supremum is taken over the family of all decreasing sequences (W_n) of clopen subsets of $\beta_o X$ with $\bigcap W_n \subset D$.

Theorem 5.11 *For an $m \in M(X)$, the following are equivalent:*

(1) $m \in M_\sigma(X)$.

(2) $|m^{\beta_o}|_*(\beta_o X \setminus \upsilon_o X) = 0$.

Proof: (1) $\Rightarrow$ (2): Let (W_n) be a decreasing sequence of clopen subsets of $\beta_o X$, with $\bigcap W_n \subset \beta_o X \setminus \upsilon_o X = 0$, and let $V_n = W_n \cap X$. Then $V_n \downarrow \emptyset$. Given $\epsilon > 0$, there exists n such that $|m|)(V_n) < \epsilon$. If W is a clopen subset of $\beta_o X$ contained in W_n, then $|m_o^\beta(W)| = |m(W \cap X)| < \epsilon$, and so $|m^{\beta_o}|(W_n) \leq \epsilon$. This proves that $|m^{\beta_o}|_*(\beta_o X \setminus \upsilon_o X) = 0$.
(2) $\Rightarrow$ (1) : Let (V_n) be a sequence of clopen subsets of X, which decreases to the emptyset, and let $W_n = \overline{V_n}^{\beta_o X}$. Then $\bigcap W_n$ is disjoint from $\upsilon_o X$ and so $\inf_n |m^{\beta_o}|(W_n) = 0$. Given $\epsilon > 0$, there exists n_o such that $|m^{\beta_o}|(W_{n_o}) \leq \epsilon$. Now, for $n \geq n_o$, we have $|m(V_n)| = |m^{\beta_o}(W_n)| < \epsilon$, and so $\lim m(V_n) = 0$, which proves that m is σ-additive. This completes the proof.

References

[1] J. Aguayo, N de Grande-de Kimpe and S. Navarro, *Strict locally convex topologies on $BC(X, \mathbb{K})$*, in: P-adic Functional Analysis, edided by W. H. Schikhof, C. Perez-Garcia and J. Kakol, Lecture Notes in Pure and Applied Mathematics, vol. **192**, Marcel Dekker, New York (1997), 1-9.

[2] J. Aguayo, N de Grande-de Kimpe and S. Navarro, *Zero-dimensional pseudocompact and ultraparacompact spaces* , in: P-adic Functional Analysis, edided by W. H. Schikhof, C. Perez- Garcia and J. Kakol, Lecture Notes in Pure and Applied Mathematics, vol. **192**, Marcel Dekker, New York (1997), 11-17.

[3] J. Aguayo, N de Grande-de Kimpe and S. Navarro, *Strict topologies and duals in spaces of functions* , in: P-adic Functional Analysis, edided by J. Kakol, N. de Grande-de Kimpe and C. Perez- Garcia , Lecture Notes in Pure and Applied Mathematics, vol. **207**, Marcel Dekker, New York (1999), 1-10.

[4] A. D. Alexandroff, *Additive set functions in abstract spaces*, a) Mat. Sb. N.S. **8**(50)(1940, 307-348, b) ibid. **9**(51)(1941), 523-628, c) ibid. **13**(55)(1943), 169-228.

[5] J. Araujo, P. Fernández-Ferreirós and J. Martinez-Maurica, *Pseudocompact and p-spaces in non-Archimedean Fuctional Analysis* , in: P-adic Functional Analysis, edided by W. H. Schikhof, N. de Grande-de Kimpe and J. Martinez-Maurica, Lecture Notes in Pure and Applied Mathematics, vol. **137**, Marcel Dekker, New York (1992), 13-21.

[6] G. Bachman, E. Beckenstein, L. Narici and S. Warner, *Rings of continuous functions with values in a topological field*, Trans. Amer. Math. Soc. **204** (1975), 91-112.

[7] N. de Grande-de Kimpe and S. Navarro, *Non-Archimedean nuclearity and spaces of continuous functions* , Indag. Math., N.S. **2** (2) (1991), 201-206.

[8] A. K. Katsaras, *Duals of non-Archimedean vector-valued function spaces*, Bull. Greek Math. Soc. **22** (1981), 25-43.

[9] A. K. Katsaras, *The strict topology in non-Archimedean vector-valued function spaces*, Proc. Kon. Ned. Akad. Wet. A **87**(2) (1984), 189-201

[10] A. K. Katsaras, *Strict topologies in non-Archimedean function spaces*, Intern. J. Math. and Math. Sci. **7**(1) (1984), 23-33.

[11] A. K. Katsaras, *On the strict topology in non-Archimedean spaces of continuous functions*, Glasnik Mat. Vol. **35** (55) (2000), 283-305.

[12] A. K. Katsaras, *Separable measures and strict topologies on spaces of non-Archimedean valued functions*, in: P-adic Numbers in Number Theory, Analytic Geometry and Functional Analysis, edided by S. Caenepeel, Bull. Belgian Math. Soc.(2002), 117-139.

[13] A. K. Katsaras, *Strict topologies and vector-measures on non-Archimedean spaces*, Cont. Math. Vol. **319** (2003), 109-129.

[14] A. K. Katsaras and A. Beloyiannis, *Tensor products of non-Archimedean weighted spaces of continuous functions*, Georgian J. Math. Vol. **6**, No 1(1999), 33-44.

[15] A. K. Katsaras and V. Benekas, *On weighted inductive limits of non-Archimedean spaces of continuous functions*, Boll. U.M.I. (**8**) 3-B(2000), 757-774.

[16] C. Perez-Garcia, *P-adic Ascoli theorems and compactoid polynomials*, Indag. Math., N. S. **3**(2) (1993), 203-210.

[17] W. H. Schikhof, *Locally convex spaces over non-spherically complete fields I, II*, Bull. Soc. Math. Belg., Ser. B, **38** (1986), 187-224

[18] A. C. M. van Rooij, Non-Archimedean Functional Analysis, New York and Basel, Marcel Dekker, 1978.

[19] R. F. Wheeler, *A survey of Baire measures and strict topologies*, Expo. Math. **2**(1983), 97-190.

Departamento de Matemática, Universidad de Concepción, Concepción, Chile
e-mail : jaguayo@udec.cl

Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
e-mail : akatsar@cc.uoi.gr

Departamento de Matemática, Universidad de Santiago, Santiago, Chile.
e-mail : snavarro@fermat.usach.cl

LARGE TIME ASYMPTOTIC TO POLYNOMIALS SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS

CH. G. PHILOS, I. K. PURNARAS AND P. CH. TSAMATOS

ABSTRACT. This article is concerned with solutions that behave asymptotically like polynomials for n-th order ($n > 1$) nonlinear ordinary differential equations. For each given integer m with $1 \leq m \leq n-1$, sufficient conditions are presented in order that, for any real polynomial of degree at most m, there exists a solution which is asymptotic at ∞ to this polynomial. Conditions are also given, which are sufficient for every solution to be asymptotic at ∞ to a real polynomial of degree at most $n-1$. The application of the results obtained to the special case of second order nonlinear differential equations leads to improved versions of the ones contained in the recent paper by Lipovan [*Glasg. Math. J.* 45 (2003), 179-187] and of other related results existing in the literature.

1. INTRODUCTION

In the asymptotic theory of n-th order ($n > 1$) nonlinear differential equations, an interesting problem is that of the study of solutions with prescribed asymptotic behavior via solutions of the equation $x^{(n)} = 0$. This problem has been extensively investigated during the last four decades for the case of second order nonlinear differential equations; see Cohen [3], Constantin [4], Dannan [7], Hallam [8], Lipovan [12], Mustafa and Rogovchenko [14], Naito [15, 16, 17], Philos and Purnaras [21], Rogovchenko and Rogovchenko [25, 26], Rogovchenko [27], Rogovchenko and Villari [28], Tong [31], Yin [33], and Zhao [34] (and the references cited in these papers). For the case of linear second order differential equations, we restrict ourselves to mention the paper by Trench [32]. The above mentioned problem has also been treated for higher order nonlinear differential equations by several researchers; see Kusano and Trench [9, 10], Meng [13], Philos [18, 19, 20], Philos, Sficas and Staikos [22], Philos and Staikos [23], and the references therein. Note that the papers [18, 19, 20, 22, 23] are concerned with differential equations with deviating arguments (including the ordinary differential equations as a particular case). We also mention here the paper by Philos and Tsamatos [24] concerning nonlinear retarded differential systems. In the present paper, we are concerned with n-th order ($n > 1$) nonlinear ordinary differential equations and we study solutions that approach real polynomials of degree at most $n-1$. Our work is essentially motivated by the recent one by Lipovan [12] for the special case of second order nonlinear ordinary differential equations; the results in [12] are extended and improved in our paper.

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Consider the *n-th order* $(n > 1)$ nonlinear differential equation

$$x^{(n)}(t) = f(t, x(t)), \quad t \geq t_0 > 0, \tag{E}$$

where *f is a continuous real-valued function on* $[t_0, \infty) \times \mathbf{R}$.

Our purpose in this paper is to investigate solutions of the differential equation (E), which behave asymptotically at ∞ like real polynomials of degree at most $n-1$, i.e. like solutions of the equation $x^{(n)} = 0$. More precisely, for each given integer m with $1 \leq m \leq n-1$, we establish sufficient conditions in order that, for any real polynomial of degree at most m, (E) has a solution defined for all large t, which is asymptotic at ∞ to this polynomial and such that the first $n-1$ derivatives of the solution are asymptotic at ∞ to the corresponding first $n-1$ derivatives of the given polynomial. We also provide conditions, which guarantee that every solution defined for all large t of (E) is asymptotic at ∞ to a real polynomial of degree at most $n-1$ (depending on the solution) and, in addition, the first $n-1$ derivatives of the solution are asymptotic at ∞ to the corresponding first $n-1$ derivatives of this polynomial. Moreover, we give sufficient conditions for every solution x defined for all large t of (E) to satisfy $x^{(n-1)}(t) \to c$ for $t \to \infty$ (and so $[x(t)/t^{n-1}] \to [c/(n-1)!]$ for $t \to \infty$), where c is some real number (depending on the solution x).

Our main results are stated in Section 2. This section contains also the application of the main results to the special case of the *second order* nonlinear differential equation

$$x''(t) = f(t, x(t)), \quad t \geq t_0 > 0. \tag{E_0}$$

The proofs of the main results are given in Section 3. Two general examples (Examples 1 and 2) are contained in the last section (Section 4). Example 1 is concerned with the application of the main results to n-th order $(n > 1)$ Emden-Fowler equations, while Example 2 illustrates the applicability of our first main result to a specific second order superlinear Emden-Fowler equation.

We note, here, that the application of our main results to the second order nonlinear differential equation (E_0) leads to improved versions of the ones given recently by Lipovan [12] (and of other previous related results in the literature) as well as to a result due to Rogovchenko and Rogovchenko [25] (see, also, Mustafa and Rogovchenko [14]).

It is an open problem to extend the results of the present paper for the more general case of n-th order $(n > 1)$ nonlinear differential equations of the form

$$x^{(n)}(t) = F(t, x(t), x'(t), ..., x^{(n-1)}(t)), \quad t \geq t_0 > 0,$$

where F is a continuous real-valued function on $[t_0, \infty) \times \mathbf{R}^n$. This problem remains interesting still in the special case of second order nonlinear differential equations of the form

$$x''(t) = F_0(t, x(t), x'(t)), \quad t \geq t_0 > 0,$$

where F_0 is a continuous real-valued function on $[t_0, \infty) \times \mathbf{R}^2$.

2. STATEMENT OF THE RESULTS

Our main results are formulated as two theorems (Theorems 1 and 2 below), a corollary of the first of these theorems, and a proposition. Our proposition plays an important role in proving the second theorem (Theorem 2); however, it is also interesting of its own as a new result.

Throughout the paper, we are interested in solutions of the differential equation (E) which are defined for all large t, i.e. in solutions of (E) on an interval $[T, \infty)$, where $T \geq t_0$ may depend on the solution. For questions about the global existence in the future of the solutions of (E), we refer to standard classical theorems in the literature (see, for example, Corduneanu [5], Cronin [6], and Lakshmikantham and Leela [11]).

Theorem 1. *Let m be an integer with $1 \leq m \leq n-1$, and assume that*

$$|f(t,z)| \leq p(t)g\left(\frac{|z|}{t^m}\right) + q(t) \quad \textit{for all } (t,z) \in [t_0, \infty) \times \mathbf{R}, \tag{2.1}$$

where p and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} t^{n-1}p(t)dt < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} t^{n-1}q(t)dt < \infty, \tag{2.2}$$

and g is a nonnegative continuous real-valued function on $[0, \infty)$ which is not identically zero.

Let $c_0, c_1, ..., c_m$ be real numbers and T be a point with $T \geq t_0$, and suppose that there exists a positive constant K so that

$$\left[\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!}p(s)ds\right] \sup\left\{g(z) : \ 0 \leq z \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right\} + \int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!}q(s)ds \leq K. \tag{2.3}$$

Then the differential equation (E) has a solution x on the interval $[T, \infty)$, which is asymptotic to the polynomial $c_0 + c_1 t + ... + c_m t^m$ for $t \to \infty$, i.e.

$$x(t) = c_0 + c_1 t + ... + c_m t^m + o(1) \quad \textit{for } t \to \infty, \tag{2.4}$$

and, in addition, satisfies

$$x^{(j)}(t) = \sum_{i=j}^{m} i(i-1)...(i-j+1)c_i t^{i-j} + o(1) \quad \textit{for } t \to \infty \qquad (j = 1, ..., m) \tag{2.5}$$

and, provided that $m < n-1$,

$$x^{(k)}(t) = o(1) \quad \textit{for } t \to \infty \qquad (k = m+1, ..., n-1). \tag{2.6}$$

Corollary. *Let m be an integer with $1 \leq m \leq n-1$, and assume that (2.1) is satisfied, where p and q, and g are as in Theorem 1.*

Then, for any real numbers $c_0, c_1, ..., c_m$, the differential equation (E) has a solution x on an interval $[T, \infty)$ (where $T \geq \max\{t_0, 1\}$ depends on $c_0, c_1, ..., c_m$), *which is asymptotic to the polynomial* $c_0 + c_1 t + ... + c_m t^m$ *for* $t \to \infty$, *i.e.* (2.4) *holds, and, in addition, satisfies* (2.5) *and (provided that* $m < n-1$) (2.6).

Proposition. *Assume that*

$$|f(t,z)| \le p(t) g\left(\frac{|z|}{t^{n-1}}\right) + q(t) \quad \text{for all } (t,z) \in [t_0, \infty) \times \mathbf{R}, \tag{2.7}$$

where p *and* q *are nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} p(t)dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)dt < \infty, \tag{2.8}$$

and g *is a continuous real-valued function on* $[0, \infty)$, *which is positive and increasing on* $(0, \infty)$ *and such that*

$$\int_1^{\infty} \frac{dz}{g(z)} = \infty. \tag{2.9}$$

Then every solution x *on an interval* $[T, \infty)$, $T \ge t_0$, *of the differential equation* (E) *satisfies*

$$x^{(n-1)}(t) = c + o(1) \quad \text{for } t \to \infty \tag{2.10}$$

and

$$x(t) = \frac{c}{(n-1)!} t^{n-1} + o(t^{n-1}) \quad \text{for } t \to \infty, \tag{2.11}$$

where c *is some real number (depending on the solution* x).

Theorem 2. *Assume that* (2.7) *is satisfied, where* p *and* q *are as in Theorem 1, and* g *is as in Proposition.*

Then every solution x *on an interval* $[T, \infty)$, $T \ge t_0$, *of the differential equation* (E) *is asymptotic to a polynomial* $c_0 + c_1 t + ... + c_{n-1} t^{n-1}$ *for* $t \to \infty$, *i.e.*

$$x(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1} + o(1) \quad \text{for } t \to \infty, \tag{2.12}$$

and, in addition, satisfies

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)...(i-j+1) c_i t^{i-j} + o(1) \quad \text{for } t \to \infty \qquad (j = 1, ..., n-1), \tag{2.13}$$

where $c_0, c_1, ..., c_{n-1}$ *are real numbers (depending on the solution* x). *More precisely, every solution* x *on an interval* $[T, \infty)$, $T \ge t_0$, *of* (E) *satisfies*

$$x(t) = C_0 + C_1(t-T) + ... + C_{n-1}(t-T)^{n-1} + o(1) \quad \text{for } t \to \infty \tag{2.14}$$

and, in addition,

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)...(i-j+1) C_i (t-T)^{i-j} + o(1) \quad \text{for } t \to \infty \qquad (j = 1, ..., n-1), \tag{2.15}$$

where

$$C_i = \frac{1}{i!}\left[x^{(i)}(T) + (-1)^{n-1-i}\int_T^{\infty}\frac{(s-T)^{n-1-i}}{(n-1-i)!}f(s,x(s))ds\right] \tag{2.16}$$
$$(i = 0, 1, ..., n-1).$$

A combination of the corollary and Theorem 2 leads to the following result:

Assume that (2.7) *is satisfied, where* p *and* q *are nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that* (2.2) *holds, and* g *is a nonnegative continuous real-valued function on* $[0, \infty)$ *which is not identically zero. Then, for any real polynomial of degree at most* $n-1$, *the differential equation* (E) *has a solution defined for all large* t, *which is asymptotic at* ∞ *to this polynomial. Moreover, if, in addition,* g *is positive and increasing on* $(0, \infty)$ *and such that* (2.9) *holds, then every solution defined for all large* t *of* (E) *is asymptotic at* ∞ *to a real polynomial of degree at most* $n-1$ (*depending on the solution*).

Now, let us concentrate on the special case of the *second order* nonlinear differential equation (E_0). In this case, Theorem 1, the corollary, the proposition, and Theorem 2 are formulated as follows:

Theorem 1A. *Assume that*

$$|f(t,z)| \leq p(t)g\left(\frac{|z|}{t}\right) + q(t) \quad \text{for all } (t,z) \in [t_0,\infty)\times \mathbf{R}, \tag{2.17}$$

where p *and* q *are nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} tp(t)dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} tq(t)dt < \infty,$$

and g *is a nonnegative continuous real-valued function on* $[0, \infty)$ *which is not identically zero.*

Let c_0, c_1 *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$\left[\int_T^{\infty}(s-T)p(s)ds\right]\sup\left\{g(z) : \ 0 \leq z \leq \frac{K}{T} + \frac{|c_0|}{T} + |c_1|\right\}$$
$$+ \int_T^{\infty}(s-T)q(s)ds \leq K.$$

Then the differential equation (E_0) *has a solution* x *on the interval* $[T, \infty)$, *which is asymptotic to the line* $c_0 + c_1 t$ *for* $t \to \infty$, *i.e.*

$$x(t) = c_0 + c_1 t + o(1) \quad \text{for } t \to \infty, \tag{2.18}$$

and, in addition, satisfies

$$x'(t) = c_1 + o(1) \quad \text{for } t \to \infty. \tag{2.19}$$

Corollary A. *Assume that* (2.17) *is satisfied, where* p *and* q, *and* g *are as in Theorem* 1A.

Then, for any real numbers c_0, c_1, *the differential equation* (E_0) *has a solution* x *on an interval* $[T, \infty)$ (*where* $T \geq \max\{t_0, 1\}$ *depends on* c_0, c_1), *which is asymptotic to the line* $c_0 + c_1 t$ *for* $t \to \infty$, *i.e.* (2.18) *holds, and, in addition, satisfies* (2.19).

Proposition A. *Assume that* (2.17) *is satisfied, where* p *and* q *are nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that* (2.8) *holds, i.e. such that*

$$\int_{t_0}^{\infty} p(t)dt < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} q(t)dt < \infty,$$

and g *is a continuous real-valued function on* $[0, \infty)$, *which is positive and increasing on* $(0, \infty)$ *and such that* (2.9) *holds, i.e. such that*

$$\int_1^{\infty} \frac{dz}{g(z)} = \infty.$$

Then every solution x *on an interval* $[T, \infty)$, $T \geq t_0$, *of the differential equation* (E_0) *satisfies*

$$x'(t) = c + o(1) \quad \textit{for } t \to \infty$$

and

$$x(t) = ct + o(t) \quad \textit{for } t \to \infty,$$

where c *is some real number* (*depending on the solution* x).

Theorem 2A. *Assume that* (2.17) *is satisfied, where* p *and* q *are as in Theorem* 1A, *and* g *is as in Proposition* A.

Then every solution x *on an interval* $[T, \infty)$, $T \geq t_0$, *of the differential equation* (E_0) *is asymptotic to a line* $c_0 + c_1 t$ *for* $t \to \infty$, *i.e.* (2.18) *holds, and, in addition, satisfies* (2.19), *where* c_0, c_1 *are real numbers* (*depending on the solution* x). *More precisely, every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (E_0) *satisfies*

$$x(t) = C_0 + C_1(t - T) + o(1) \quad \textit{for } t \to \infty$$

and, in addition,

$$x'(t) = C_1 + o(1) \quad \textit{for } t \to \infty,$$

where

$$C_0 = x(T) - \int_T^{\infty} (s - T) f(s, x(s))ds \quad \textit{and} \quad C_1 = x'(T) + \int_T^{\infty} f(s, x(s))ds.$$

The main results in the recent paper by Lipovan [12] are formulated as two theorems (Theorems 1 and 2). Theorem 1 in [12] is contained in Corollary A, while Theorem 2 in [12] is included in Theorem 2A. Note, also, that Proposition A has been previously established by Rogovchenko and Rogovchenko [25] (see, also, Mustafa and Rogovchenko [14]).

3. PROOFS OF THE MAIN RESULTS

In order to prove Theorem 1, we will apply the fixed point technique, by using the following Schauder's theorem (see Schauder [29]).

The Schauder theorem. *Let E be a Banach space and X any nonempty convex and closed subset of E. If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point (i.e. there exists an $x \in X$ with $x = Sx$).*

In the proof of Theorem 1, we use the Schauder theorem with $E = B([T, \infty))$, where $B([T, \infty))$ is the Banach space of all continuous and bounded real-valued functions on the interval $[T, \infty)$ endowed with the sup-norm $\|\cdot\|$:

$$\|h\| = \sup_{t \geq T} |h(t)| \quad \text{for } h \in B([T, \infty)).$$

We need the following compactness criterion for subsets of $B([T, \infty))$, which is a corollary of the Arzelà-Ascoli theorem (see Avramescu [1]; see, also, Staikos [30]).

Compactness criterion. *Let H be an equicontinuous and uniformly bounded subset of the Banach space $B([T, \infty))$. If H is equiconvergent at ∞, it is also relatively compact.*

Note that a set H of real-valued functions defined on the interval $[T, \infty)$ is called *equiconvergent at ∞* if all functions in H are convergent in $\mathbf{R}$ at the point ∞ and, in addition, for every $\epsilon > 0$ there exists a $T' \geq T$ such that, for all functions h in H, it holds

$$\left| h(t) - \lim_{s \to \infty} h(s) \right| < \epsilon \quad \text{for all } t \geq T'.$$

To prove our proposition we will make use of the well-known Bihari's lemma (see Bihari [2]; see, also, Corduneanu [5]). This lemma is given here in a simple form which suffices for our needs.

The Bihari lemma. *Assume that*

$$h(t) \leq M + \int_{T_0}^{t} \mu(s) g(h(s)) ds \quad \textit{for } t \geq T_0,$$

where M is a positive constant, h and μ are nonnegative continuous real-valued functions on $[T_0, \infty)$, and g is a continuous real-valued function on $[0, \infty)$, which is positive and increasing on $(0, \infty)$ and such that

$$\int_1^{\infty} \frac{dz}{g(z)} = \infty.$$

Then

$$h(t) \leq G^{-1} \left(G(M) + \int_{T_0}^{t} \mu(s) ds \right) \quad \textit{for } t \geq T_0,$$

where G is a primitive of $1/g$ on $(0, \infty)$ and G^{-1} is the inverse function of G.

Now, we are in the position to proceed with the proofs of our main results.

Proof of Theorem 1. The substitution

$$y(t) = x(t) - (c_0 + c_1 t + ... + c_m t^m)$$

transforms the differential equation (E) into the equation

$$(\mathrm{E}^*) \qquad y^{(n)}(t) = f\left(t, y(t) + \sum_{i=0}^{m} c_i t^i\right), \quad t \geq t_0 > 0.$$

We immediately see that

$$y^{(j)}(t) = x^{(j)}(t) - \sum_{i=j}^{m} i(i-1)...(i-j+1)c_i t^{i-j} \quad (j = 1, ..., m)$$

and, provided that $m < n-1$,

$$y^{(k)}(t) = x^{(k)}(t) \quad (k = m+1, ..., n-1).$$

Hence, by taking into account (2.4), (2.5) and (2.6), we conclude that all we have to prove is that the differential equation (E^*) has a solution y on the interval $[T, \infty)$ with

$$(3.1) \qquad \lim_{t\to\infty} y^{(\rho)}(t) = 0 \quad (\rho = 0, 1, ..., n-1).$$

Consider the Banach space $E = B([T, \infty))$ endowed with the sup-norm $\|\cdot\|$, and define

$$Y = \{y \in E: \ \|y\| \leq K\}.$$

Clearly, Y is a nonempty convex and closed subset of E.

Now, let y be an arbitrary function in Y. For every $t \geq T$, we have

$$\frac{\left|y(t) + \sum_{i=0}^{m} c_i t^i\right|}{t^m} \leq \frac{|y(t)|}{t^m} + \sum_{i=0}^{m} \frac{|c_i|}{t^{m-i}} \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}.$$

Consequently

$$g\left(\frac{\left|y(t) + \sum_{i=0}^{m} c_i t^i\right|}{t^m}\right) \leq \Theta \quad \text{for every } t \geq T,$$

where

$$\Theta \equiv \Theta(c_0, c_1, ..., c_m; T; K) = \sup\left\{g(z): \ 0 \leq z \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right\}.$$

On the other hand, (2.1) gives

$$\left|f\left(t, y(t) + \sum_{i=0}^{m} c_i t^i\right)\right| \leq p(t) g\left(\frac{\left|y(t) + \sum_{i=0}^{m} c_i t^i\right|}{t^m}\right) + q(t) \quad \text{for } t \geq T.$$

So, it follows that

$$(3.2) \qquad \left|f\left(t, y(t) + \sum_{i=0}^{m} c_i t^i\right)\right| \leq \Theta p(t) + q(t) \quad \text{for all } t \geq T.$$

Thus, in view of (2.2), we conclude that

$$\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \quad \text{exists in } \mathbf{R}$$

and, more generally,

$$\int_T^\infty \frac{(s-T)^{n-1-\rho}}{(n-1-\rho)!} f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right) ds \quad \text{exists in } \mathbf{R} \qquad (\rho = 0, 1, ..., n-1).$$

Furthermore, by using (3.2), we obtain for every $t \geq T$,

$$\begin{aligned}
&\left| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right) ds \right| \\
&\leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right)\right| ds \\
&\leq \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} \left| f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right)\right| ds \\
&\leq \Theta \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} p(s) ds + \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} q(s) ds.
\end{aligned}$$

Hence, by taking into account (2.3), we have

$$\left| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right) ds \right| \leq K \quad \text{for every } t \geq T. \tag{3.3}$$

As (3.3) is true for any function $y \in Y$, we can immediately conclude that the formula

$$(Sy)(t) = (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right) ds \quad \text{for } t \geq T$$

defines a mapping S of Y into itself. We shall prove that this mapping satisfies the assumptions of the Schauder theorem.

First, we will show that SY is relatively compact. Since $SY \subseteq Y$, it follows immediately that SY is uniformly bounded. Moreover, for each function y in Y, we can use (3.2) to derive for all $t \geq T$

$$\begin{aligned}
|(Sy)(t) - 0| &= \left| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right) ds \right| \\
&\leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| f\left(s, y(s)+\sum_{i=0}^m c_i s^i\right)\right| ds \\
&\leq \Theta \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s) ds + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s) ds.
\end{aligned}$$

So, by taking into account (2.2), we can easily verify that SY is equiconvergent at ∞. Furthermore, by using again (3.2), for any $y \in Y$ and every t_1, t_2 with

$T \le t_1 < t_2$, we get

$$
\begin{aligned}
|(Sy)(t_2) - (Sy)(t_1)| &= \left| \int_{t_2}^{\infty} \frac{(s-t_2)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right. \\
&\qquad \left. - \int_{t_1}^{\infty} \frac{(s-t_1)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right| \\
&= \left| \int_{t_2}^{\infty} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right] dr \right. \\
&\qquad \left. - \int_{t_1}^{\infty} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right] dr \right| \\
&= \left| - \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right] dr \right| \\
&\le \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} \left| f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) \right| ds \right] dr \\
&\le \Theta \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} p(s) ds \right] dr \\
&\qquad + \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} q(s) ds \right] dr.
\end{aligned}
$$

Thus, because of (2.2), it follows that SY is equicontinuous. By the given compactness criterion, SY is relatively compact.

Next, we shall prove that the mapping S is continuous. Let $y \in Y$ and $(y_\nu)_{\nu \ge 1}$ be an arbitrary sequence in Y with

$$\lim_{\nu \to \infty} y_\nu = y.$$

By (3.2), we have

$$\left| f\left(t, y_\nu(t) + \sum_{i=0}^{m} c_i t^i\right) \right| \le \Theta p(t) + q(t) \quad \text{for every } t \ge T \text{ and for all } \nu \ge 1$$

and hence, by taking into account (2.2), we can apply the Lebesgue dominated convergence theorem to obtain, for each $t \ge T$,

$$
\begin{aligned}
&\lim_{\nu \to \infty} \int_{t}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y_\nu(s) + \sum_{i=0}^{m} c_i s^i\right) ds \\
&\qquad = \int_{t}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds.
\end{aligned}
$$

So, we have the pointwise convergence

$$\lim_{\nu \to \infty} (Sy_\nu)(t) = (Sy)(t) \quad \text{for } t \ge T.$$

It remains to verify that the convergence is also uniform, i.e.

$$\lim_{\nu \to \infty} Sy_\nu = Sy. \tag{3.4}$$

To this end, let us consider an arbitrary subsequence $(Sy_{\mu_\nu})_{\nu\geq 1}$ of $(Sy_\nu)_{\nu\geq 1}$. Since SY is relatively compact, there exists a subsequence $\left(Sy_{\mu_{\lambda_\nu}}\right)_{\nu\geq 1}$ of $(Sy_{\mu_\nu})_{\nu\geq 1}$ and a $u\in E$ so that

$$\lim_{\nu\to\infty} Sy_{\mu_{\lambda_\nu}} = u.$$

As the uniform convergence implies the pointwise convergence to the same limit function, we always have $u = Sy$. We have thus proved that (3.4) holds. Consequently, S is continuous.

Finally, the Schauder theorem implies that there exists a $y\in Y$ with $y = Sy$, i.e.

$$y(t) = (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds \quad \text{for every } t\geq T.$$

Then we immediately obtain

$$y^{(n)}(t) = f\left(t, y(t) + \sum_{i=0}^m c_i t^i\right) \quad \text{for all } t\geq T,$$

which means that y is a solution on the interval $[T,\infty)$ of the differential equation (E*). We also have

$$y^{(\rho)}(t) = (-1)^{n-\rho} \int_t^\infty \frac{(s-t)^{n-1-\rho}}{(n-1-\rho)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds \quad \text{for all } t\geq T$$

$$(\rho = 0,1,\dots,n-1)$$

and consequently the solution y satisfies (3.1).

The proof of the theorem is complete.

Proof of the corollary. Let $c_0, c_1, \dots, c_m$ be given real numbers. Consider a positive constant K so that

$$\Theta_0 \equiv \sup\left\{g(z):\ 0\leq z\leq K + \sum_{i=0}^m |c_i|\right\} > 0.$$

(Such a K exists because of the hypothesis that g is not identically zero on $[0,\infty)$.) By (2.2), we can choose a point $T\geq \max\{t_0, 1\}$ such that

$$\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} p(s)ds \leq \frac{K}{2\Theta_0} \quad \text{and} \quad \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} q(s)ds \leq \frac{K}{2}.$$

Since $T\geq 1$, we have

$$\frac{K}{T^m} + \sum_{i=0}^m \frac{|c_i|}{T^{m-i}} \leq K + \sum_{i=0}^m |c_i|$$

and consequently

$$\Theta \equiv \sup\left\{g(z):\ 0\leq z\leq \frac{K}{T^m} + \sum_{i=0}^m \frac{|c_i|}{T^{m-i}}\right\}$$
$$\leq \sup\left\{g(z):\ 0\leq z\leq K + \sum_{i=0}^m |c_i|\right\} \equiv \Theta_0.$$

Thus, we obtain

$$\left[\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!}p(s)ds\right]\Theta + \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!}q(s)ds \leq \frac{K}{2\Theta_0}\Theta + \frac{K}{2} \leq K,$$

i.e. (2.3) is satisfied. So, the corollary follows immediately from Theorem 1.

Proof of the proposition. Let x be a solution on an interval $[T,\infty)$, $T \geq t_0$, of the differential equation (E). Then (E) gives

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!}x^{(i)}(T) + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!}f(s,x(s))ds \quad \text{for } t \geq T.$$

Thus, by using (2.7), we obtain for every $t \geq T$

$$\begin{aligned}
|x(t)| &\leq \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!}\left|x^{(i)}(T)\right| + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!}\left|f(s,x(s))\right| ds \\
&\leq \sum_{i=0}^{n-1} \frac{t^i}{i!}\left|x^{(i)}(T)\right| + t^{n-1}\int_T^t \frac{1}{(n-1)!}\left[p(s)g\left(\frac{|x(s)|}{s^{n-1}}\right) + q(s)\right] ds \\
&\leq \left[\sum_{i=0}^{n-1} \frac{t^i}{i!}\left|x^{(i)}(T)\right| + \frac{t^{n-1}}{(n-1)!}\int_T^\infty q(s)ds\right] + t^{n-1}\int_T^t \frac{p(s)}{(n-1)!}g\left(\frac{|x(s)|}{s^{n-1}}\right) ds.
\end{aligned}$$

So, we have

$$\frac{|x(t)|}{t^{n-1}} \leq \left[\sum_{i=0}^{n-1} \frac{1}{i!t^{n-1-i}}\left|x^{(i)}(T)\right| + \frac{1}{(n-1)!}\int_T^\infty q(s)ds\right] + \int_T^t \frac{p(s)}{(n-1)!}g\left(\frac{|x(s)|}{s^{n-1}}\right) ds$$

for all $t \geq T$. Thus, because of the second assumption of (2.8), there exists a positive constant M so that

$$\frac{|x(t)|}{t^{n-1}} \leq M + \int_T^t \frac{p(s)}{(n-1)!}g\left(\frac{|x(s)|}{s^{n-1}}\right) ds \quad \text{for every } t \geq T. \tag{3.5}$$

Next, we define

$$G(z) = \int_M^z \frac{du}{g(u)} \quad \text{for } z \geq M.$$

(Note that $g(u) > 0$ for $u \geq M > 0$.) Clearly, G is a primitive of the function $1/g$ on $[M,\infty)$. We observe that $G(M) = 0$ and that G is strictly increasing on $[M,\infty)$. Moreover, we see that (2.9) implies $G(\infty) = \infty$. Thus, the range of G is equal to $[0,\infty)$. Let G^{-1} be the inverse function of G. The function G^{-1} is also strictly increasing on its domain $[0,\infty)$, and the range of G^{-1} equals to $[M,\infty)$. In view of the above observations, we can take into account (3.5) and use the Bihari lemma to obtain

$$\frac{|x(t)|}{t^{n-1}} \leq G^{-1}\left(G(M) + \int_T^t \frac{p(s)}{(n-1)!}ds\right) = G^{-1}\left(\frac{1}{(n-1)!}\int_T^t p(s)ds\right) \quad \text{for } t \geq T.$$

Hence, by taking into account the first assumption of (2.8), we get

$$\frac{|x(t)|}{t^{n-1}} \leq G^{-1}\left(\frac{1}{(n-1)!}\int_T^\infty p(s)ds\right) \quad \text{for every } t \geq T,$$

i.e. there exists a positive constant N so that

$$\frac{|x(t)|}{t^{n-1}} \leq N \quad \text{for all } t \geq T. \tag{3.6}$$

Now, by using (2.7) and (3.6), we derive

$$|f(t,x(t))| \leq p(t)g\left(\frac{|x(t)|}{t^{n-1}}\right) + q(t) \leq p(t) \sup_{0\leq z\leq N} g(z) + q(t) \quad \text{for } t \geq T.$$

Thus, because of (2.8), it follows immediately that

$$\int_T^\infty f(s,x(s))ds \quad \text{exists (as a real number).}$$

But, (E) gives

$$x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^t f(s,x(s))ds \quad \text{for } t \geq T.$$

Therefore,

$$\lim_{t\to\infty} x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^\infty f(s,x(s))ds \equiv c \in \mathbf{R},$$

i.e. (2.10) holds. Finally, by the L' Hospital rule, we obtain

$$\lim_{t\to\infty} \frac{x(t)}{t^{n-1}} = \frac{1}{(n-1)!} \lim_{t\to\infty} x^{(n-1)}(t) = \frac{c}{(n-1)!}$$

and consequently the solution x satisfies (2.11).

The proof of the proposition has been completed.

Proof of Theorem 2. Let x be a solution on an interval $[T,\infty)$, $T \geq t_0$, of the differential equation (E). We observe that (2.2) implies (2.8). Thus, as in the proof of the proposition, we conclude that there exists a positive constant N such that (3.6) holds. (Note that this conclusion can be obtained from the proposition itself, since it guarantees that $\lim_{t\to\infty} \left[x(t)/t^{n-1}\right] = C$ for some real number C.) By virtue of (2.7) and (3.6), we have

$$|f(t,x(t))| \leq p(t)g\left(\frac{|x(t)|}{t^{n-1}}\right) + q(t) \leq p(t) \sup_{0\leq z\leq N} g(z) + q(t) \quad \text{for } t \geq T.$$

So, by taking into account (2.2), we see that

$$L_i \equiv \int_T^\infty \frac{(s-T)^{n-1-i}}{(n-1-i)!} f(s,x(s))ds \quad (i = 0,1,...,n-1)$$

are real numbers.

Now, from (E) it follows that

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} x^{(i)}(T) + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \quad \text{for } t \geq T. \tag{3.7}$$

For every $t \geq T$, we obtain

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s))ds \\
&= -\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} d\left[\int_s^\infty f(r, x(r))dr\right] \\
&= \frac{(t-T)^{n-1}}{(n-1)!} \int_T^\infty f(r, x(r))dr - \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} \left[\int_s^\infty f(r, x(r))dr\right] ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} \left[\int_s^\infty f(r, x(r))dr\right] ds.
\end{aligned}$$

Let us assume that $n > 2$. Then we derive, for $t \geq T$,

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s))ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} + \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} d\left[\int_s^\infty (r-s) f(r, x(r))dr\right] \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \frac{(t-T)^{n-2}}{(n-2)!} \int_T^\infty (r-T) f(r, x(r))dr \\
&\qquad + \int_T^t \frac{(t-s)^{n-3}}{(n-3)!} \left[\int_s^\infty (r-s) f(r, x(r))dr\right] ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \frac{(t-T)^{n-2}}{(n-2)!} L_{n-2} \\
&\qquad + \int_T^t \frac{(t-s)^{n-3}}{(n-3)!} \left[\int_s^\infty (r-s) f(r, x(r))dr\right] ds.
\end{aligned}$$

If $n > 3$, then we can apply the same arguments to obtain, for every $t \geq T$,

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s))ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \frac{(t-T)^{n-2}}{(n-2)!} L_{n-2} + \frac{(t-T)^{n-3}}{(n-3)!} L_{n-3} \\
&\qquad - \int_T^t \frac{(t-s)^{n-4}}{(n-4)!} \left[\int_s^\infty \frac{(r-s)^2}{2!} f(r, x(r))dr\right] ds.
\end{aligned}$$

Following this procedure, we finally conclude that

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s))ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} (-1)^0 L_{n-1} + \frac{(t-T)^{n-2}}{(n-2)!} (-1)^1 L_{n-2} + \ldots + \frac{(t-T)^2}{2!} (-1)^{n-3} L_2 \\
&\qquad + \frac{(t-T)^1}{1!} (-1)^{n-2} L_1 + (-1)^{n-1} \int_T^t \left[\int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} f(r, x(r))dr\right] ds
\end{aligned}$$

for all $t \geq T$. Furthermore, we have for $t \geq T$

$$\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds$$

$$= \sum_{i=1}^{n-1} \frac{(t-T)^i}{i!}(-1)^{n-1-i}L_i + (-1)^{n-1}\int_T^\infty \left[\int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} f(r,x(r))dr\right] ds$$

$$- (-1)^{n-1}\int_t^\infty \left[\int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} f(r,x(r))dr\right] ds$$

$$= \sum_{i=1}^{n-1} \frac{(t-T)^i}{i!}(-1)^{n-1-i}L_i + (-1)^{n-1}\int_T^\infty \frac{(r-T)^{n-1}}{(n-1)!} f(r,x(r))dr$$

$$+ (-1)^n \int_t^\infty \frac{(r-t)^{n-1}}{(n-1)!} f(r,x(r))dr$$

$$= \sum_{i=1}^{n-1} \frac{(t-T)^i}{i!}(-1)^{n-1-i}L_i + (-1)^{n-1}L_0 + (-1)^n \int_t^\infty \frac{(r-t)^{n-1}}{(n-1)!} f(r,x(r))dr$$

$$= \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!}(-1)^{n-1-i}L_i + (-1)^n \int_t^\infty \frac{(r-t)^{n-1}}{(n-1)!} f(r,x(r))dr.$$

Thus, (3.7) yields

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!}\left[x^{(i)}(T) + (-1)^{n-1-i}L_i\right] + (-1)^n \int_t^\infty \frac{(r-t)^{n-1}}{(n-1)!} f(r,x(r))dr$$

for all $t \geq T$. Hence, by taking into account the definition of L_i $(i = 0, 1, ..., n-1)$ as well as (2.16), we see that

$$x(t) = \sum_{i=0}^{n-1} C_i(t-T)^i + (-1)^n \int_t^\infty \frac{(r-t)^{n-1}}{(n-1)!} f(r,x(r))dr \quad \text{for all } t \geq T. \tag{3.8}$$

Since

$$\lim_{t\to\infty} \int_t^\infty \frac{(r-t)^{n-1}}{(n-1)!} f(r,x(r))dr = 0,$$

it follows from (3.8) that the solution x satisfies (2.14). Moreover, from (3.8) we obtain

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)...(i-j+1)C_i(t-T)^{i-j} \tag{3.9}$$

$$+ (-1)^{n-j}\int_t^\infty \frac{(r-t)^{n-1-j}}{(n-1-j)!} f(r,x(r))dr \quad \text{for } t \geq T \qquad (j = 1, ..., n-1).$$

Thus, in view of the fact that

$$\lim_{t\to\infty} \int_t^\infty \frac{(r-t)^{n-1-j}}{(n-1-j)!} f(r,x(r))dr = 0 \qquad (j = 1, ..., n-1),$$

(3.9) guarantees that the solution x satisfies also (2.15). Finally, it is clear that there exist real numbers $c_0, c_1, ..., c_{n-1}$ so that

$$C_0 + C_1(t-T) + ... + C_{n-1}(t-T)^{n-1} \equiv c_0 + c_1 t + ... + c_{n-1}t^{n-1}$$

and so x satisfies (2.12) and, in addition, (2.13).

The proof of the theorem is now complete.

4. EXAMPLES

Example 1. Consider the n-th order ($n > 1$) Emden-Fowler equation

$$x^{(n)}(t) = a(t)\,|x(t)|^{\gamma}\,\mathrm{sgn}x(t), \quad t \geq t_0 > 0, \tag{D}$$

where a *is a continuous real-valued function on* $[t_0, \infty)$ *and* γ *is a positive real number.*

An application of Theorem 1 to the differential equation (D) leads to the following result: *Let* m *be an integer with* $1 \leq m \leq n-1$, *and assume that*

$$\int_{t_0}^{\infty} t^{n-1+m\gamma}\,|a(t)|\,dt < \infty. \tag{4.1}$$

Let $c_0, c_1, ..., c_m$ *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$\left[\int_{T}^{\infty} \frac{(s-T)^{n-1}}{(n-1)!} s^{m\gamma}\,|a(s)|\,ds\right]\left(\frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right)^{\gamma} \leq K.$$

Then (D) *has a solution* x *on the interval* $[T, \infty)$ *with the property:* (P(x)) x *is asymptotic to the polynomial* $c_0 + c_1 t + ... + c_m t^m$ *for* $t \to \infty$, *i.e.* (2.4) *holds, and, in addition, it satisfies* (2.5) *and* (*provided that* $m < n-1$) (2.6).

Also, by applying the corollary to the differential equation (D), we arrive at the next result: *Let* m *be an integer with* $1 \leq m \leq n-1$, *and assume that* (4.1) *is satisfied. Then, for any real numbers* $c_0, c_1, ..., c_m$, (D) *has a solution* x *on an interval* $[T, \infty)$ (*where* $T \geq \max\{t_0, 1\}$ *depends on* $c_0, c_1, ..., c_m$) *with the property* (P(x)).

Moreover, we can apply the proposition for the differential equation (D) to obtain the result: *If*

$$\int_{t_0}^{\infty} t^{(n-1)\gamma}\,|a(t)|\,dt < \infty \tag{4.2}$$

and $\gamma \leq 1$, *then every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (D) *satisfies* (2.10) *and* (2.11), *where* c *is some real number* (*depending on the solution* x).

Furthermore, by an application of Theorem 2 to the differential equation (D), we can be led to the following result: *Assume that*

$$\int_{t_0}^{\infty} t^{(n-1)(1+\gamma)}\,|a(t)|\,dt < \infty \tag{4.3}$$

and that $\gamma \leq 1$. *Then every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (D) *is asymptotic to a polynomial* $c_0 + c_1 t + ... + c_{n-1} t^{n-1}$ *for* $t \to \infty$, *i.e.* (2.12) *holds, and, in addition, satisfies* (2.13), *where* $c_0, c_1, ..., c_{n-1}$ *are real numbers* (*depending on the solution* x). *More precisely, every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (D) *satisfies* (2.14) *and, in addition,* (2.15), *where*

$$C_i = \frac{1}{i!}\left[x^{(i)}(T) + (-1)^{n-1-i} \int_{T}^{\infty} \frac{(s-T)^{n-1-i}}{(n-1-i)!} a(s)\,|x(s)|^{\gamma}\,\mathrm{sgn}x(s)ds\right]$$

$$(i = 0, 1, ..., n-1).$$

Now, let us consider the particular case of the Emden-Fowler equation (D) with

$$a(t) = t^{\lambda}\mu(t) \quad \text{for } t \geq t_0,$$

where λ is a real number and μ is a continuous and bounded real-valued function on $[t_0, \infty)$. In this case, we have

$$|a(t)| \leq \theta t^{\lambda} \quad \text{for every } t \geq t_0,$$

where θ is a positive constant. We immediately see that (4.1) is satisfied if $\lambda < -(n + m\gamma)$. Moreover, we observe that (4.2) holds if $\lambda < -[1 + (n-1)\gamma]$, while (4.3) is fulfilled if $\lambda < -[1 + (n-1)(1+\gamma)]$.

Example 2. Consider the second order superlinear Emden-Fowler equation

$$\text{(d)} \qquad x''(t) = a(t)[x(t)]^2 \mathrm{sgn}x(t), \quad t \geq t_0 > 0,$$

where a *is a continuous real-valued function on* $[t_0, \infty)$.

By applying Theorem 1 (or, in particular, Theorem 1A) to the differential equation (d), we are led to the following result: *Assume that*

$$\text{(4.4)} \qquad \int_{t_0}^{\infty} t^3 |a(t)|\, dt < \infty.$$

Let c_0, c_1 *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$\text{(4.5)} \qquad A(T)\left(\frac{K}{T} + \frac{|c_0|}{T} + |c_1|\right)^2 \leq K,$$

where

$$\text{(4.6)} \qquad A(T) = \int_T^{\infty} (s-T)s^2 |a(s)|\, ds.$$

Then (d) *has a solution* x *on the interval* $[T, \infty)$, *which is asymptotic to the line* $c_0 + c_1 t$ *for* $t \to \infty$, *i.e.*

$$\text{(4.7)} \qquad x(t) = c_0 + c_1 t + o(1) \quad \textit{for } t \to \infty,$$

and, in addition, satisfies

$$\text{(4.8)} \qquad x'(t) = c_1 + o(1) \quad \textit{for } t \to \infty.$$

Now, assume that (4.4) is satisfied, and let c_0, c_1 be given real numbers and $T \geq t_0$ be a fixed point. Moreover, let $A(T)$ be defined by (4.6). In the trivial case where $A(T) = 0$, (4.5) holds by itself for any positive constant K. So, in what follows, it will be supposed that $A(T) > 0$. We easily verify that, for every positive constant K, (4.5) can equivalently be written as

$$\text{(4.9)} \qquad K^2 + 2\left[(|c_0| + |c_1|T) - \frac{T^2}{2A(T)}\right]K + (|c_0| + |c_1|T)^2 \leq 0.$$

Consider the quadratic equation

$$\Omega(\omega) \equiv \omega^2 + 2\left[(|c_0| + |c_1|T) - \frac{T^2}{2A(T)}\right]\omega + (|c_0| + |c_1|T)^2 = 0$$

in the complex plane, and let Δ be its discriminant, i.e.

$$\Delta = \left\{2\left[(|c_0| + |c_1|T) - \frac{T^2}{2A(T)}\right]\right\}^2 - 4(|c_0| + |c_1|T)^2.$$

We immediately find

$$\Delta = 4\frac{T^2}{A(T)}\left[-(|c_0| + |c_1|T) + \frac{T^2}{4A(T)}\right].$$

In the case where $\Delta < 0$, the equation $\Omega(\omega) = 0$ has no real roots and consequently $\Omega(\omega) > 0$ for all $\omega \in \mathbf{R}$. Thus, in this case, there is no positive constant K so that (4.9) is fulfilled. If $\Delta = 0$, i.e.

$$|c_0| + |c_1|T = \frac{T^2}{4A(T)}, \tag{4.10}$$

then the equation $\Omega(\omega) = 0$ has exactly one (double) real root ω_0 given by

$$\omega_0 = \frac{T^2}{4A(T)}.$$

Hence, in case that (4.10) holds, (4.9) is fulfilled (as an equality) for $K = \omega_0 > 0$, i.e. there exists a positive constant K so that (4.9) is satisfied. Next, let us consider the case where $\Delta > 0$, i.e.

$$|c_0| + |c_1|T < \frac{T^2}{4A(T)}. \tag{4.11}$$

Then the equation $\Omega(\omega) = 0$ has the real roots

$$\omega_1 = -\left(|c_0| + |c_1|T\right) + \frac{T^2}{2A(T)} - \sqrt{\frac{T^2}{A(T)}\left[-(|c_0| + |c_1|T) + \frac{T^2}{4A(T)}\right]}$$

and

$$\omega_2 = -\left(|c_0| + |c_1|T\right) + \frac{T^2}{2A(T)} + \sqrt{\frac{T^2}{A(T)}\left[-(|c_0| + |c_1|T) + \frac{T^2}{4A(T)}\right]}$$

with $\omega_1 < \omega_2$. For each real number ω, it holds

$$\Omega(\omega) \leq 0 \quad \text{if and only if} \quad \omega_1 \leq \omega \leq \omega_2.$$

By (4.11), we have

$$-\left(|c_0| + |c_1|T\right) + \frac{T^2}{2A(T)} > -\left(|c_0| + |c_1|T\right) + \frac{T^2}{4A(T)} > 0$$

and consequently ω_2 is positive. Therefore,

$$\Omega(\omega) \leq 0 \quad \text{for any } \omega \in (\max\{0, \omega_1\}, \omega_2].$$

So, provided that (4.11) is satisfied, there exists a positive constant K so that (4.9) holds. We have thus proved that there exists a positive constant K so that (4.9) (or, equivalently, (4.5)) is satisfied if and only if either (4.10) or (4.11) is fulfilled, i.e. if and only if

$$|c_0| + |c_1|T \leq \frac{T^2}{4A(T)},$$

which can equivalenlty be written as

$$A(T)\left(|c_0| + |c_1|T\right) \leq \frac{T^2}{4}. \tag{4.12}$$

We observe that (4.12) is also true if $A(T) = 0$. After the above, we can have the following result:

Assume that (4.4) *is satisfied, and let* c_0, c_1 *be real numbers and* $T \geq t_0$ *be a point so that* (4.12) *holds, where* $A(T)$ *is defined by* (4.6). *Then* (d) *has a solution* x *on the interval* $[T, \infty)$, *which satisfies* (4.7) *and* (4.8).

Finally, let us consider the particular case of the Emden-Fowler equation (d) with

$$a(t) = t^{\lambda}\mu(t) \quad \text{for } t \geq t_0,$$

where λ is a real number and μ is a continuous and bounded real-valued function on $[t_0, \infty)$. In this case, there exists a positive constant θ so that

$$|a(t)| \leq \theta t^{\lambda} \quad \text{for every } t \geq t_0.$$

We immediately see that (4.4) is satisfied if $\lambda < -4$. Furthermore, assume that $\lambda < -4$ and let c_0, c_1 be real numbers and $T \geq t_0$ be a point. Here, we have

$$A(T) \equiv \int_T^{\infty} (s-T)s^2 \, |a(s)| \, ds \leq \theta \int_T^{\infty} (s-T)s^{\lambda+2} ds = \theta \frac{T^{\lambda+4}}{(\lambda+3)(\lambda+4)}.$$

So, (4.12) is satisfied if

$$T^{\lambda+2} \left(|c_0| + |c_1| \, T\right) \leq \frac{(\lambda+3)(\lambda+4)}{4\theta}.$$

REFERENCES

[1] C. AVRAMESCU, Sur l' existence des solutions convergentès de systèmes d' équations différentielles non linéaires, *Ann. Mat. Pura Appl.* **81** (1969), 147-168.

[2] I. BIHARI, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hungar.* **7** (1956), 81-94.

[3] D. S. COHEN, The asymptotic behavior of a class of nonlinear differential equations, *Proc. Amer. Math. Soc.* **18** (1967), 607-609.

[4] A. CONSTANTIN, On the asymptotic behavior of second order nonlinear differential equations, *Rend. Mat. Appl.* **13** (1993), 627-634.

[5] C. CORDUNEANU, *Principles of Differential and Integral Equations*, Chelsea Publishing Company, The Bronx, New York, 1977.

[6] J. CRONIN, *Differential Equations: Introduction and Qualitative Theory*, Second Edition, Revised and Expanded, Marcel Dekker, Inc., New York, 1994.

[7] F. M. DANNAN, Integral inequalities of Gronwall-Bellman-Bihari type and asymptotic behavior of certain second order nonlinear differential equations, *J. Math. Anal. Appl.* **108** (1985), 151-164.

[8] T. G. HALLAM, Asymptotic integration of second order differential equations with integrable coefficients, *SIAM J. Appl. Math.* **19** (1970), 430-439.

[9] T. KUSANO AND W. F. TRENCH, Global existence theorems for solutions of nonlinear differential equations with prescribed asymptotic behavior, *J. London Math. Soc.* **31** (1985), 478-486.

[10] T. KUSANO AND W. F. TRENCH, Existence of global solutions with prescribed asymptotic behavior for nonlinear ordinary differential equations , *Ann. Mat. Pura Appl.* **142** (1985), 381-392.

[11] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities, Vol. I*, Academic Press, New York, 1969.

[12] O. LIPOVAN, On the asymptotic behaviour of the solutions to a class of second order nonlinear differential equations, *Glasg. Math. J.* **45** (2003), 179-187.

[13] F. W. MENG, A note on Tong paper: The asymptotic behavior of a class of nonlinear differential equations of second order, *Proc. Amer. Math. Soc.* **108** (1990), 383-386.

[14] O. G. MUSTAFA AND Y. V. ROGOVCHENKO, Global existence of solutions with prescribed asymptotic behavior for second-order nonlinear differential equations, *Nonlinear Anal.* **51** (2002), 339-368.

[15] M. NAITO, Asymptotic behavior of solutions of second order differential equations with integrable coefficients, *Trans. Amer. Math. Soc.* **282** (1984), 577-588.

[16] M. NAITO, Nonoscillatory solutions of second order differential equations with integrable coefficients, *Proc. Amer. Math. Soc.* **109** (1990), 769-774.

[17] M. NAITO, Integral averages and the asymptotic behavior of solutions of second order ordinary differential equations, *J. Math. Anal. Appl.* **164** (1992), 370-380.

[18] CH. G. PHILOS, Oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments, *Hiroshima Math. J.* **8** (1978), 31-48.

[19] CH. G. PHILOS, On the oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments, *Ann. Mat. Pura Appl.* **119** (1979), 25-40.

[20] CH. G. PHILOS, Asymptotic behaviour of a class of nonoscillatory solutions of differential equations with deviating arguments, *Math. Slovaca* **33** (1983), 409-428.

[21] CH. G. PHILOS AND I. K. PURNARAS, Asymptotic behavior of solutions of second order nonlinear ordinary differential equations, *Nonlinear Anal.* **24** (1995), 81-90.

[22] CH. G. PHILOS, Y. G. SFICAS AND V. A. STAIKOS, Some results on the asymptotic behavior of nonoscillatory solutions of differential equations with deviating arguments, *J. Austral. Math. Soc. Series A* **32** (1982), 295-317.

[23] CH. G. PHILOS AND V. A. STAIKOS, A basic asymptotic criterion for differential equations with deviating arguments and its applications to the nonoscillation of linear ordinary equations, *Nonlinear Anal.* **6** (1982), 1095-1113.

[24] CH. G. PHILOS AND P. CH. TSAMATOS, Asymptotic equilibrium of retarded differential equations, *Funkcial. Ekvac.* **26** (1983), 281-293.

[25] S. P. ROGOVCHENKO AND Y. V. ROGOVCHENKO, Asymptotic behavior of solutions of second order nonlinear differential equations, *Portugal. Math.* **57** (2000), 17-33.

[26] S. P. ROGOVCHENKO AND Y. V. ROGOVCHENKO, Asymptotic behavior of certain second order nonlinear differential equations, *Dynam. Systems Appl.* **10** (2001), 185–200.

[27] Y. V. ROGOVCHENKO, On the asymptotic behavior of solutions for a class of second order nonlinear differential equations, *Collect. Math.* **49** (1998), 113-120.

[28] Y. V. ROGOVCHENKO AND G. VILLARI, Asymptotic behaviour of solutions for second order nonlinear autonomous differential equations, *NoDEA Nonlinear Differential Equations Appl.* **4** (1997), 271-282.

[29] J. SCHAUDER, Der Fixpuktsatz in Funktionalräumen, *Studia Math.* **2** (1930), 171-180.

[30] V. A. STAIKOS, *Differential Equations with Deviating Arguments - Oscillation Theory,* Unpublished manuscripts.

[31] J. TONG, The asymptotic behavior of a class of nonlinear differential equations of second order, *Proc. Amer. Math. Soc.* **84** (1982), 235-236.

[32] W. F. TRENCH, On the asymptotic behavior of solutions of second order linear differential equations, *Proc. Amer. Math. Soc.* **14** (1963), 12-14.

[33] Z. YIN, Monotone positive solutions of second-order nonlinear differential equations, *Nonlinear Anal.* **54** (2003), 391-403.

[34] Z. ZHAO, Positive solutions of nonlinear second order ordinary differential equations, *Proc. Amer. Math. Soc.* **121** (1994), 465-469.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, P. O. BOX 1186, 451 10 IOANNINA, GREECE

E-mail address: cphilos@cc.uoi.gr ; ipurnara@cc.uoi.gr ; ptsamato@cc.uoi.gr

The Use of Inverse Deformation Mapping in Mesh Optimization

Vassilios K. Kalpakides[1] and Konstantinos G. Balassas

Department of Mathematics, University of Ioannina, Ioannina, GR-45110, Greece

Abstract

This paper aims at the exploitation of the material forces to find an optimum mesh in the finite element method. The classical variational formulation provides the linear momentum equation in Lagrangian description. A variational setting for the derivation of the canonical momentum equation in Eulerian description is presented. The latter is based on an extremum principle for the total potential energy functional defined in terms of the inverse deformation function. This constitutes a theoretical framework which allows the formulation of the finite element method for the canonical momentum equation as well as the computation of the material forces arising from the discretization. Thus, apart from the finite element solution for the standard boundary value problem of elastostatics, a second one for the canonical momentum equation can be formulated and solved numerically. The former provides an optimum deformation by minimizing the standard total potential energy, namely solving the physical forces equilibrium equation. The latter provides an optimum discretization by minimizing the total potential energy in terms of the inverse deformation function, that is solving the material force equilibrium equation. The theoretical considerations are supported by providing a computational example.

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1 Introduction

In elasticity the motion of the body is governed by the linear momentum equation which expresses either the balance of momentum in elastodynamics or the equilibrium of the forces acting on the body in elastostatics. The contributors to the linear momentum equation are the classical forces (either body or contact). The material forces [7, 8, 5] are contributors to another equation, the so called material or canonical momentum equation. In statics, this equation governs the equilibrium of the material forces. Unlike the problems with defects where the notion of material force takes a physical interpretation, in the framework of pure elasticity the canonical momentum equation, being a simple identity for the solution of the linear momentum equation, does not bring any new information to the problem of motion or equilibrium. Nevertheless, this is true only when the accurate solution is concerned. In computations, where approximate solutions are necessarily considered, the material forces make sense. They are closely related with the numerical error [3, 9]. More specifically, a function, being an approximate solution of the momentum equation,

[1] Corresponding author. E-mail address: vkalpak@cc.uoi.gr

does not necessarily fulfil the canonical momentum equation. The departure of the latter from zero – that is a non vanishing material force – actually measures the numerical error, for the specific approximate solution, inserted due to the specific choice of discretization. One can try to minimize these material forces by displacing appropriately the nodes points of the mesh [1, 11, 12, 13, 18].

Theoretically, the material forces, even those arising from the discretization, are governed by the material momentum equation. Hence, the right way to reduce them is to solve the canonical momentum equation. In a way, this can be done by the Arbitrary Lagrangian Eulerian (ALE) formulation of the finite element method [2]. For more details about the connection of ALE formulation with the concept of material forces see in [6, 19]. This approach is based on an additional configuration, the so called mesh configuration, on which both the spatial and material variables are interpolated. Thus, one obtains an optimum solution over an optimum mesh as well.

Unlike the ALE formulation, where essentially the momentum and canonical momentum equations are solved simultaneously, we propose a scheme where these equations are solved successively. Thus, no need for a new configuration exists. To this end, we use the momentum equation in Lagrangian description and the canonical momentum equation in Eulerian description. Thus, the FE solution of the linear momentum equation provides automatically the discretization for the solution of canonical momentum equation and vice versa. This can be repeated so that an iteration computational scheme can be established. It is remarked that even if the momentum equation is a linear one, the corresponding canonical momentum equation is non–linear making the problem of mesh optimization more complicated in comparison with the initial problem of finding the optimum approximate deformation for a given mesh. From the theoretical point of view, the scheme is supported by a variational principle of the total potential energy functional expressed in terms of the inverse deformation function. This principle results in the canonical momentum equation as a necessary condition for an extremum of the energy functional. It also provides the, needed for the FE implementation, weak form of the canonical momentum equation.

Although, in this paper we confine ourselves to elastostatics, we shall use the terms linear momentum and canonical momentum equations instead of the equilibrium of physical forces and equilibrium of material forces equations, respectively. Two distinct symbols ∇_R and ∇ are used to denote the gradient with respect to material $\mathbf{X}$ and spatial $\mathbf{x}$ variables, respectively. In the same spirit the denotations Grad and grad as well as Div and div will be used. Also, the reader must pay attention to the difference between the total differential operator $d/d\mathbf{X}$ and the simple partial differential operator $\partial/\partial\mathbf{X}$. The same difference exists between the differential operators $d/d\mathbf{x}$ and $\partial/\partial\mathbf{x}$.

In Section 2, the direct and inverse problem of elastostatics as well as the strong and weak forms of the corresponding equations are presented. In Section 3, the role of material forces in FEM is examined and the justification of the proposed computational scheme is exposed. The FE implementation is presented in Section 4 and some numerical results are derived in Section 5.

2 The formulation in terms of the inverse deformation function

2.1 Direct deformation function. Preliminaries

We recall first the standard notions of kinematics denoting with B_R the reference configuration and B the deformed configuration in equilibrium. Also, denoting with $\mathbf{f}$ the deformation mapping, we write

$$\mathbf{f} : \mathsf{B}_R \to \mathsf{B}, \quad \mathbf{x} = \mathbf{f}(\mathbf{X}), \quad \mathsf{B}_R, \mathsf{B} \subset E^3, \tag{2.1}$$

thus, $\mathbf{f}$ maps the material particle X_A from the referential domain B_R to its spatial position $x_i = f_i(X_A)$ in the spatial domain B. As usual, we require f to be one-to-one and sufficiently smooth (typically C^2 for classical elasticity). Its gradient, $\nabla_R \mathbf{f}$ is denoted by $\mathbf{F}$ ($F_{iA} = \partial f_i / \partial X_A$) and its corresponding Jacobian, $\det(\mathbf{F})$ by J. Certainly under the above requirements, $J \neq 0$, for all $\mathbf{X} \in \mathsf{B}_R$.

Let us assume that the body forces per unit volume, $\mathbf{b}(\mathbf{x})$ ($\mathbf{b} = b_i$) are conservative, that is independent of the deformation path. Denoting by $V(\mathbf{x})$ the corresponding potential energy density, one can write

$$\mathbf{b} = -\frac{\partial V}{\partial \mathbf{x}} = -\mathrm{grad} V. \tag{2.2}$$

So, the total potential energy density becomes

$$U = U(\mathbf{X}, \mathbf{f}, \mathbf{F}) = W(\mathbf{X}, \mathbf{F}) + V(\mathbf{f}), \tag{2.3}$$

where W is the stored energy density per unit referential volume. From now on, we restrict ourselves to the case of Dirichlet boundary conditions, let

$$\mathbf{f}(\mathbf{X}) = \mathbf{h}(\mathbf{X}), \text{ for all } \mathbf{X} \in \partial \mathsf{B}_R,$$

where $\mathbf{h}(\mathbf{X})$, $\mathbf{X} \in \partial \mathsf{B}_R$, is a given function. The total potential energy of the whole body for any admissible deformation $\mathbf{f}$ is given by the functional

$$\mathbf{I}[\mathbf{f}] = \int_{\mathsf{B}_R} U(\mathbf{X}, \mathbf{f}, \mathbf{F}) dX = \int_{\mathsf{B}_R} [W(\mathbf{X}, \mathbf{F}) + V(\mathbf{f})] dX, \ \mathbf{f} \in \mathcal{V}, \tag{2.4}$$

where $\mathcal{V}$ is the set of admissible functions for $\mathbf{I}$, that is, all sufficiently smooth (for instance C^2) functions defined on B_R, which moreover fulfil the boundary conditions of the problem.

The necessary condition for an extremum in $\mathcal{V}$ for the total potential energy (2.4) provides the Euler–Lagrange equations

$$\delta \mathbf{I}[\mathbf{f}; \delta \mathbf{f}] = 0, \ \forall \ \delta \mathbf{f} \in \mathcal{V}_0 \ \Rightarrow \frac{\partial U}{\partial \mathbf{f}} - \frac{d}{d\mathbf{X}} \left(\frac{\partial U}{\partial \mathbf{F}} \right) = 0, \quad \forall \ \mathbf{X} \in \mathsf{B}_R,$$

where $\mathcal{V}_0$ is the set of admissible variations, that is all C^1 functions defined on B_R which, moreover, fulfil homogeneous conditions along the boundary $\partial \mathsf{B}_R$.

Accounting now for the constitutive relations of hyperelasticity ($\mathbf{T}^T = \partial W/\partial \mathbf{F}$) and eq. (2.2), we can write the Euler–Lagrange equations in the form

$$\mathrm{Div}\mathbf{T}^T + \mathbf{b} = 0, \quad \forall\, \mathbf{X} \in \mathsf{B}_R, \tag{2.5}$$

where $\mathbf{T}$ is the first Piola-Kirchhoff stress tensor. Eq. (2.5) is the strong form of the momentum equation in the referential description. For completeness, we give the corresponding one in the spatial description

$$\mathrm{div}\boldsymbol{\sigma}^T + J^{-1}\mathbf{b} = 0, \quad \forall\, \mathrm{x} \in \mathsf{B}, \tag{2.6}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor related to the Piola-Kirchhoff stress tensor as follows

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{T} \quad \text{and} \quad \mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{2.7}$$

Remark 2.1 In the subsequent sections it will be crucial to distinguish the domain over which any function (consequently any equation as well) is defined. That's why the domain, over which the equations (2.5) and (2.6) hold, is specified. Moreover, the fact that some relationships contain functions defined over different domains, may cause confusion . If such is the case, one should account for the proper composition through the function $\mathbf{f}$ which relates the spatial and material variables. For instance, eq. (2.7b) should be read as

$$\mathbf{T}(\mathbf{X}) = J(\mathbf{X})\mathbf{F}^{-1}(\mathbf{f}(\mathbf{X}))\boldsymbol{\sigma}(\mathbf{f}(\mathbf{X})).$$

Let us look now for a solution among less smooth functions (let $\mathbf{f}$ be C^1), that is to enlarge the set of admissible functions denoted now by $\bar{\mathcal{V}}$. A necessary condition in order the functional $\mathbf{I}$ to possess an extremum in $\bar{\mathcal{V}}$ is given by the variational principle

$$\delta\mathbf{I}[\mathbf{f};\delta\mathbf{f}] = 0, \ \forall\ \delta\mathbf{f} \in \mathcal{V}_0 \ \Rightarrow \int_{\mathsf{B}_R} \left(\frac{\partial W}{\partial \mathbf{F}} : \delta\mathbf{F} + \frac{\partial V}{\partial \mathbf{f}} \cdot \delta\mathbf{f}\right) dX = 0, \ \forall\ \delta\mathbf{f} \in \mathcal{V}_0,$$

or

$$\int_{\mathsf{B}_R} \left(\mathbf{T}^T : \nabla_R \delta\mathbf{f} - \mathbf{b} \cdot \delta\mathbf{f}\right) dX = 0, \ \forall\ \delta\mathbf{f} \in \mathcal{V}_0. \tag{2.8}$$

The above equation is the weak form of momentum equation (2.5) or the virtual work principle for hyperelastostatics. The first term under the integral represents the internal virtual work while the second one represents the external virtual work, namely the virtual work of the external forces (here the body forces).

2.2 Inverse deformation function. A variational principle for the canonical momentum equation

Here, we are interested in the inverse deformation function, $\mathbf{g} = \mathbf{f}^{-1}$, i.e.,

$$\mathbf{g} : \mathsf{B} \to \mathsf{B}_R, \quad \mathbf{X} = \mathbf{g}(\mathrm{x}). \tag{2.9}$$

The gradient of $\mathbf{g}$ is denoted by $\mathbf{G} = \nabla\mathbf{g}$ ($G_{AI} = \partial g_A/\partial x_i$) and j denotes the corresponding Jacobian determinant $\det(\mathbf{G})$.

We can express the total potential energy of the body in terms of the inverse function of $\mathbf{f}$ [7, 10]

$$\tilde{\mathbf{I}}[\mathbf{g}] = \int_{\mathsf{B}} \tilde{U}(\mathbf{x}, \mathbf{g}, \mathbf{G})\,dx = \int_{\mathsf{B}} [\tilde{W}(\mathbf{g}, \mathbf{G}) + \tilde{V}(\mathbf{x}, \mathbf{G})]\,dx, \;\; \mathbf{g} \in \mathcal{U}, \tag{2.10}$$

where

$$\tilde{W}(\mathbf{g}, \mathbf{G}) = \jmath(\mathbf{G}) W(\mathbf{g}, \mathbf{G}^{-1}), \quad \tilde{V}(\mathbf{x}, \mathbf{G}) = \jmath(\mathbf{G}) V(\mathbf{x}), \tag{2.11}$$

and $\mathcal{U}$ is the set of admissible functions of $\tilde{\mathbf{I}}$, that is, all C^2 functions on B that fulfil the boundary conditions

$$\mathbf{g}(\mathbf{x}) = \mathbf{h}^{-1}(\mathbf{x}), \text{ for all } \mathbf{x} \in \partial\mathsf{B}. \tag{2.12}$$

Using now the relations given by eqs. (2.4), (2.10) and (2.11), one can easily confirm that

$$\mathbf{I}[\mathbf{f}] = \tilde{\mathbf{I}}[\mathbf{g}], \text{ for all } (\mathbf{f}, \mathbf{g}) \in \mathcal{V} \times \mathcal{U} \text{ and } \mathbf{g} = \mathbf{f}^{-1}. \tag{2.13}$$

Remark 2.2 It is noted that both the energy functionals $\mathbf{I}$ and $\tilde{\mathbf{I}}$ express the same physical quantity, that is the total potential energy. With the definition (2.10), one essentially transforms the domain of integration (B_R into B) as well as the domain of the energy functional ($\mathcal{V}$ into $\mathcal{U}$). For a more detailed discussion about the manipulations which lead to the definition of $\tilde{\mathbf{I}}$, we cite the Refs. [10, 14].

An extremum principle for the functional $\tilde{\mathbf{I}}$ in the general setting of thermoelastodynamics has been been given in [10]. There, it has been proved that a necessary condition for an extremum of the total energy functional, defined in terms of the inverse motion mapping, provides the canonical momentum equation for the motion. For the needs of the present paper it is enough to invoke the special case of equilibrium which can be stated as:

For an hyperelastic body loaded by conservative body forces a stationary point of the functional (2.10) satisfies the strong form of the canonical momentum equation for the equilibrium.

An extremum $\mathbf{g} \in \mathcal{U}$ will fulfil the variational equation

$$\delta\tilde{\mathbf{I}}[\mathbf{g}; \delta\mathbf{g}] = 0, \quad \forall\; \delta\mathbf{g} \in \mathcal{U}_0,$$

where $\mathcal{U}_0$ is the corresponding set of variations. The above equation leads to the Euler–Lagrange equations corresponding to the functional $\tilde{\mathbf{I}}$.

$$\frac{\partial \tilde{U}}{\partial \mathbf{g}} - \frac{d}{d\mathbf{x}}\left(\frac{\partial \tilde{U}}{\partial \mathbf{G}}\right) = 0, \quad \forall\, \mathbf{x} \in \mathsf{B}. \tag{2.14}$$

We introduce the following definitions

$$\mathbf{C}^T = \frac{\partial \tilde{W}}{\partial \mathbf{G}}, \quad \mathbf{B}^{inh} = -\frac{\partial W}{\partial \mathbf{g}}. \tag{2.15}$$

The last of eqs (2.15) is the *inhomogeneity material force*. Notice that $\partial W/\partial \mathbf{g} = \partial W/\partial \mathbf{X}$. Recalling eq. (2.11b), we directly obtain

$$\frac{\partial \tilde{V}}{\partial \mathbf{G}} = \frac{\partial j}{\partial \mathbf{G}} V = j\mathbf{F}^T V = \mathbf{F}^T \tilde{V}. \tag{2.16}$$

Accounting for the identity [7]

$$\operatorname{div}(j\mathbf{F}^T) = 0, \tag{2.17}$$

we obtain

$$\frac{d}{d\mathbf{x}}\left(\frac{\partial \tilde{V}}{\partial \mathbf{G}}\right) = \operatorname{div}(j\mathbf{F}^T V) = j\mathbf{F}^T \operatorname{grad} V = -j\mathbf{F}^T \mathbf{b}. \tag{2.18}$$

With the aid of eqs. (2.15), (2.16) and (2.18), the Euler–Lagrange equations take the form

$$\operatorname{div}\mathbf{C}^T - j\mathbf{F}^T\mathbf{b} + j\mathbf{B}^{inh} = 0, \quad \forall\, \mathbf{x} \in \mathsf{B}.$$

The above equation can be equivalently written in the following form

$$\operatorname{div}\mathbf{C}^T + j(\mathbf{B} + \mathbf{B}^{inh}) = 0, \quad \forall\, \mathbf{x} \in \mathsf{B}, \tag{2.19}$$

where

$$\mathbf{B} = \operatorname{Grad} V = \operatorname{grad} V\ \mathbf{F} = -\mathbf{F}^T\mathbf{b}. \tag{2.20}$$

Eq. (2.19) is the canonical momentum equation in spatial description. From the continuum mechanics point of view, eq. (2.19) interprets the equilibrium of the material forces like eq. (2.5), in the direct formalism, interprets the equilibrium of the physical forces.

To establish a relationship of the tensor quantity $\mathbf{C}$ with known quantities, we use the following identity [10]

$$\frac{\partial W}{\partial \mathbf{G}} = -\mathbf{F}^T \frac{\partial W}{\partial \mathbf{F}} \mathbf{F}^T.$$

Thus, we can proceed to the following simple calculation with the aid of eqs. (2.11)

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial \mathbf{G}} &= W\frac{\partial j}{\partial \mathbf{G}} + j\frac{\partial W}{\partial \mathbf{G}} = Wj\mathbf{F}^T - j\mathbf{F}^T\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T \\ &= \left(W\mathbf{I}_R - \mathbf{F}^T\frac{\partial W}{\partial \mathbf{F}}\right) j\mathbf{F}^T = \left(W\mathbf{I}_R - \mathbf{F}^T\mathbf{T}^T\right) j\mathbf{F}^T \\ &= j\mathbf{\Sigma}^T\mathbf{F}^T, \end{aligned} \tag{2.21}$$

where

$$\mathbf{\Sigma} = W\mathbf{I}_R - \mathbf{T}\mathbf{F} \tag{2.22}$$

is the well-known Eshelby stress tensor (see [4] for the linear case). Actually, eq. (2.21) reads

$$\mathbf{\Sigma} = j^{-1}\mathbf{G}\mathbf{C} \quad \text{and} \quad \mathbf{C} = j\mathbf{G}^{-1}\mathbf{\Sigma} \tag{2.23}$$

thus, the Eshelby stress tensor $\mathbf{\Sigma}$ is the Piola transformation (pull–back) of the two–point tensor $\mathbf{C}$. Recalling now eq. (2.17), one can easily calculate the divergence of (2.23b)

$$\mathrm{div}\mathbf{C}^T = j\mathrm{Div}\mathbf{\Sigma}^T, \tag{2.24}$$

thus, the pull–back of eq. (2.19) is given by the equation

$$\mathrm{Div}\mathbf{\Sigma}^T + \mathbf{B} + \mathbf{B}^{inh} = 0, \ \mathbf{X} \in \mathsf{B}_R. \tag{2.25}$$

The latter is obviously an alternative form of the canonical momentum equation. In literature, it is customarily referred to as the pseudomomentum equation [7, 8]. Both equations (2.19) and (2.25) express the equilibrium of material forces. The single difference between the two equations is that the former is set in the spatial description while the latter in the referential description.

Remark 2.3 Notice the analogy between the equations (2.5), (2.6) and (2.7) on the one hand and equations (2.19), (2.25) and (2.23) on the other. An elegant dualism between the two formalisms for the direct and inverse deformation function is so revealed. For a more thorough discussion about this we mention the recent works of Steinmann [15, 16, 17]. We would like to draw the attention on the equation of canonical momentum in the spatial description, i.e., eq. (2.19) instead of the corresponding one in the referential description (eq. (2.25)) that is commonly used in literature. This preference will be justified in the next section, where we will develop our computational scheme.

Returning now to the functional of total potential energy in terms of the inverse deformation function (i.e. eq. (2.10)), one can additionally obtain the weak form of the material momentum equation by relaxing the smoothness assumptions for the functions $\mathbf{g}$ belonging to the set $\mathcal{U}$. Let $\bar{\mathcal{U}}$ be the set of C^1 functions which satisfy the boundary conditions (2.12). Consider a $\mathbf{g} \in \bar{\mathcal{U}}$ which is a stationary point of $\tilde{\mathbf{I}}$, then we take

$$\delta\tilde{\mathbf{I}}[\mathbf{g};\delta\mathbf{g}] = 0, \ \forall\delta\mathbf{g} \in \mathcal{U}_0 \ \Leftrightarrow \ \int_{\mathsf{B}} \left[\left(\frac{\partial \tilde{W}}{\partial \mathbf{G}} + \frac{\partial \tilde{V}}{\partial \mathbf{G}} \right) : \delta\mathbf{G} + \frac{\partial \tilde{W}}{\partial \mathbf{g}} \cdot \delta\mathbf{g} \right] dx = 0, \ \forall\delta\mathbf{g} \in \mathcal{U}_0 \tag{2.26}$$

or taking into account eqs. (2.15) and (2.16) one can write

$$\int_{\mathsf{B}} \left[\left(\mathbf{C}^T + \mathbf{G}^{-T}\tilde{V} \right) : \nabla\delta\mathbf{g} - j\mathbf{B}^{inh} \cdot \delta\mathbf{g} \right] dx = 0, \ \ \forall \ \delta\mathbf{g} \in \mathcal{U}_0. \tag{2.27}$$

Equation (2.27) is the weak form of the canonical momentum equation. Its corresponding strong form is given by eq. (2.19). One can view eq. (2.27) as *the principle of virtual work of the material forces.*

Remark 2.4 It is interesting to note that the material body forces $j\mathbf{B}$ originated by the conservative physical body forces $\mathbf{b}$, do not contribute to the "external" virtual work through a term of the form $j\mathbf{B} \cdot \delta\mathbf{g}$ as one may expect. The "external" virtual work is exclusively due to the inhomogeneity material forces. As it appears in eq. (2.27), the material body forces contribute through the term $\mathbf{G}^{-T}\tilde{V} : \nabla\delta\mathbf{g}$, thus it seems they are contributors to the "internal" virtual work.

3 The material forces and the FE solution

It is worth noting that the momentum equation (2.5) as well as the material momentum equation (2.19) solve actually the same problem i.e., the equilibrium of the elastic body. The former provides the deformation function while the latter provides its inverse. If the solution $\mathbf{f}$ of the momentum equation is given, then its inverse, i.e. $\mathbf{f}^{-1}$, fulfils the material momentum equation identically and vice-versa. Thus, the material momentum equation is a simple identity for the solution of the direct problem. This is not true for any other function of the set of admissible deformations. In the sequel, we restrict ourselves to the case where $W = W(\mathbf{F})$ thus, the inhomogeneity material forces vanish. Given any function $f \in \mathcal{U}$, which is not a solution of eq. (2.5), then eq. (2.19) does not vanish for its inverse $\mathbf{g} \equiv \mathbf{f}^{-1}$, thus

$$\mathrm{div}\mathbf{C}^T(\mathbf{g}) + \jmath(\mathbf{g})\mathbf{B}(\mathbf{g}) = \mathcal{B}^c(\mathbf{g}) \neq 0. \tag{3.1}$$

The vector quantity $\mathcal{B}^c$ is a material force which indicates the deviation of $\mathbf{f}$ from the accurate solution. The smaller is the magnitude of $\mathcal{B}^c(\mathbf{g})$, the closer is $\mathbf{f}$ to the solution of momentum equation. More rigorously, any $\mathbf{g} \in \mathcal{U}$ (or, equivalently any $\mathbf{f} \in \mathcal{V}$) defines a vector field of material forces over the deformed configuration, i.e., $\mathcal{B}^c(\mathbf{g}(\mathbf{x})) = \mathbf{B}^c(\mathbf{x})$, $\mathbf{x} \in \mathsf{B}$. The presence of a non–vanishing field $\mathbf{B}^c(\mathbf{x})$ is an evidence that the body, for the specific $\mathbf{g}$, is not in equilibrium. Any attempt to reduce the material force $\mathbf{B}^c$ brings us closer to the solution. Notice that fixing the deformation $\mathbf{x}$ and letting the material points $\mathbf{X}$ free to variate, the equation $\mathbf{B}^c(\mathbf{x}) = 0$, $\mathbf{x} \in \mathsf{B}$ reduces to the canonical momentum equation (2.19). Introducing a more "physical" view, one might interpret $\mathbf{B}^c$ as the material forces field that try to "move" the material points to meet the true undeformed configuration B_R for a given deformed configuration B.

Remark 3.1 We remind that the weak form of the momentum (eq. (2.8)) and the canonical momentum equation (eq. (2.27)) have been formulated in the classical way. Notice, that the sets $\bar{\mathcal{V}}$, $\mathcal{V}_0$, $\bar{\mathcal{U}}$ and $\mathcal{U}_0$ contain generally C^1 functions. Whereas this formulation is appropriate for a virtual work principle, it is not adequate for a finite element application, because a FE solution is not in general a C^1 function. The natural setting of a variational principle is that of Sobolev spaces. In this perspective, one should replace $\bar{\mathcal{V}}$ and $\mathcal{V}_0$ with $H^1(\mathsf{B}_R)$ and $H^1_0(\mathsf{B}_R)$, respectively as well as $\bar{\mathcal{U}}$ and $\mathcal{U}_0$ with $H^1(\mathsf{B})$ and $H^1_0(\mathsf{B})$.

Consider now an arbitrary discretization of B_R with N nodes $\mathbf{X}_0^1, \mathbf{X}_0^2, \ldots, \mathbf{X}_0^N$ and the corresponding finite element solution of the momentum equation (direct problem)

$$\mathbf{f}^h = \sum_{J=1}^{N} \mathbf{x}^J N^J(\mathbf{X}), \tag{3.2}$$

where $N^J(\mathbf{X})$ are the shape functions. The corresponding total potential energy is given by

$$\mathbf{I}[\mathbf{f}^h] = \mathbf{I}_h(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) \tag{3.3}$$

and the determination of the unknown parameters of the FE solution results from the solution of the algebraic system

$$\frac{\partial \mathbf{I}_h(\mathbf{x}^I)}{\partial \mathbf{x}^J} = 0, \quad I, J = 1, \ldots, N. \tag{3.4}$$

At this point, it is worth noting that the expression (3.3) for the total energy depends tacitly on the choice of the nodes in the initial discretization of B_R, thus, we can symbolically write

$$\mathbf{I}_h(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) = E(\underbrace{\mathbf{X}_0^1, \mathbf{X}_0^2, \ldots, \mathbf{X}_0^N}_{\text{fixed}}; \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N). \tag{3.5}$$

Furthermore, let us denote by $\{\mathbf{x}_0^I,\ I = 1, \ldots, N\}$, the solution of the system (3.4), then the following inequality holds

$$\mathbf{I}_h(\mathbf{x}_0^I) = E(\mathbf{X}_0^J; \mathbf{x}_0^I) \leq E(\mathbf{X}_0^J; \mathbf{x}^I),\ \forall \mathbf{x}^I \in \mathrm{B}. \tag{3.6}$$

According to eq. (3.1), the so obtained approximate deformation does not fulfil the canonical momentum equation, i.e. $\mathcal{B}^c(\mathbf{f}^{h-1}) \neq 0$. A little bit later, in this section we will show how one can compute the corresponding discrete material forces $\mathbf{B}_I^c, I = 1, 2 \ldots, N$, which will be indicators of the induced numerical error. More precisely, the magnitude and the direction of each discrete material force is influenced by the choice of the initial mesh $\mathbf{X}_0^1, \mathbf{X}_0^2, \ldots, \mathbf{X}_0^N$ as well as by the induced deformation $\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^N$. Thus any rearrangement of the initial mesh $\{\mathbf{X}_0^J\}$ causes different material forces, which become smaller as far as the corresponding mesh provides a better approximate solution $\mathbf{f}^h$. Motivating from this remark some methods of minimizing the material forces $\mathbf{B}_I^c,\ I = 1, \ldots, N$ by displacing appropriately the nodes of the mesh have been proposed [1, 11, 12, 13, 18].

In this work, we propose an alternative way to obtain the appropriate displacement of the nodes. First, we remark that, by analogy with the continuous case, *the discrete material forces try to 'move" the initial nodes $\{\mathbf{X}_0^J,\ J = 1, \ldots, N\}$ to meet the appropriate mesh that fits better to the discrete deformed configuration* $\{\mathbf{x}_0^I,\ I = 1, \ldots, N\}$. This is equivalent to solving the canonical momentum equation (inverse problem) for the deformed configuration $\{\mathbf{x}_0^I\}$ be given or, in mathematical terms, to determine the FE solution

$$\mathbf{g}^h = \sum_{J=1}^{N} \mathbf{X}^J M^J(\mathbf{x}), \tag{3.7}$$

where $M^J(\mathbf{x})$ are now the shape functions arisen from the discretization $\{\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^N\}$ of B. To obtain the unknown parameters of $\mathbf{g}^h$, which, by the way, will not be the inverse of $\mathbf{f}^h$, one should minimize

$$\tilde{\mathbf{I}}[\mathbf{g}^h] = \tilde{\mathbf{I}}_h(\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^N) = E(\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^N; \underbrace{\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^N}_{\text{fixed}}) \tag{3.8}$$

or, equivalently

$$\frac{\partial \tilde{\mathbf{I}}_h(\mathbf{X}^I)}{\partial \mathbf{X}^J} = 0, \quad I, J = 1, \ldots, N. \tag{3.9}$$

Eqs. (3.9) constitute the finite dimensional version (discrete equations) of the variational equation (2.27). Computing the left hand side of the above equations for the specific values of the initial discretization $\{\mathbf{X}_0^I\}$, we obtain the *discrete material forces* $\mathbf{B}_J^c$ *at the nodes* x_0^J *of* B.

$$\frac{\partial \tilde{\mathbf{I}}_h}{\partial \mathbf{X}^J}(\mathbf{X}_0^I) = \mathbf{B}_J^c, \quad J = 1, \ldots, N.$$

Notice that the above equation is the discrete version of eq. (3.1). Also, the material forces so computed are defined on the deformed configuration unlike what is commonly done in literature, where the discrete material forces are computed on the reference configuration. Minimizing the material forces $\mathbf{B}_J^c$ is equivalent to solving the system (3.9). The latter is always a non-linear algebraic system because the stored energy density $\tilde{W}$, from which it is arising, is not a quadratic function even if the W is. Let $\{\mathbf{X}_1^I,\ I = 1, \ldots, N\}$ be the solution of the system (3.9), then we can write

$$\tilde{\mathbf{I}}_h(\mathbf{X}_1^J) = E(\mathbf{X}_1^J; \mathrm{x}_0^I) \leq E(\mathbf{X}^J; \mathrm{x}_0^I), \quad \forall \mathbf{X}^J \in \mathsf{B}_R. \tag{3.10}$$

Combining the inequalities. (3.6) and (3.10), we obtain

$$E(\mathbf{X}_1^J; \mathrm{x}_0^I) \leq E(\mathbf{X}_0^J; \mathrm{x}_0^I), \tag{3.11}$$

thus, the deformation $\{(\mathbf{X}_1^I; \mathrm{x}_0^I),\ I = 1, \ldots, N\}$ represents a better approximation to the accurate solution in comparison with the deformation $\{(\mathbf{X}_0^I; \mathrm{x}_0^I),\ I = 1, \ldots, N\}$. Taking the view of Braun [3], we can claim that the discetization of B_R, $\{\mathbf{X}_1^I,\ I = 1, \ldots, N\}$ is more appropriate than the initial, arbitrarily chosen discitization $\{\mathbf{X}_0^I,\ I = 1, \ldots, N\}$. However, we must underline the fact that $\{\mathbf{X}_1^I\}$ make up the optimum discretization under the constraint that the points $\{\mathrm{x}_0^I\}$ are fixed. To obtain a better approximation one has to compute the FE solution of the momentum equation for the new discretiztion $\{\mathbf{X}_1^I\}$. The above considerations make sure that its solution $\{\mathrm{x}_1^I\}$ will provide lower energy, namely it will be an even better approximation. This completes a series of computations which can be repeated establishing, in this way, an iterative computational scheme. Its k step gives the deformation $\{(\mathbf{X}_k^I; \mathrm{x}_k^I),\ I = 1, \ldots, N\}$ which has lower energy in comparison with the $k-1$ step. Thus, the relaxation of energy at every step ensures the convergence of the scheme.

4 Implementation of the FE method

We start with an arbitrary discretization in the reference configuration, B_R according to the standard FEM. Let $\{\mathbf{X}_0^I,\ \ I = 1, 2, \ldots, N\}$ be the nodes in B_R which set up the N_{el} elements Ω_e, i.e.,

$$\mathsf{B}_R = \bigcup_{e=1}^{N_{el}} \Omega_e. \tag{4.1}$$

Denoting with $N_e^I(\mathrm{x})$ the shape functions for the element e, one can write the finite element solution for this element as

$$\mathbf{f}^e(\mathbf{X}) = \sum_{I=1}^{m} \mathrm{x}^I N_e^I(\mathbf{X}), \tag{4.2}$$

where $\mathbf{x}^I$ are the unknown nodal parameters and m is the number of nodes per element. The same shape functions can be used also to interpolate the variations $\delta\mathbf{f}$

$$\delta\mathbf{f}^e(\mathbf{X}) = \sum_{I=1}^{m} \delta\mathbf{x}^I N_e^I(\mathbf{X}). \tag{4.3}$$

The corresponding gradients will be given by

$$\nabla_R \mathbf{f}^e = \sum_{I=1}^{m} \mathbf{x}^I \otimes \nabla_R N_e^I, \quad \nabla_R \delta\mathbf{f}^e = \sum_{I=1}^{m} \delta\mathbf{x}^I \otimes \nabla_R N_e^I. \tag{4.4}$$

Thus, following the standard procedure one obtains the momentum equation in discrete form

$$R_f^I(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N) = \mathop{\mathrm{A}}_{e=1}^{n_{el}} r_{e,f}^I = 0, \quad I = 1, \ldots, N, \tag{4.5}$$

where n_{el} is the number of neighbouring elements to the node I, N is the total number of nodes, $\mathop{\mathrm{A}}\limits_{e=1}^{n_{el}}$ denotes the assembly of all element contributions at the node I and

$$r_{e,f}^I = \int_{\Omega_e} (\nabla_R N_e^I(\mathbf{X}) \cdot \mathbf{T} - N_e^I(\mathbf{X})\ \mathbf{b}) dX \tag{4.6}$$

is the residual at the node I of the element e. We remind that the above algebraic system corresponds to a Dirichlet problem, thus only the parameters $\mathbf{x}^I$ at the interior nodes should be determined. The values at the boundary nodes are given by the essential boundary conditions. If the system (4.5) is a linear one, a standard procedure [2], which accounts for the insertion of the boundary values into the system, can be followed. In the case it is a non-linear one, one should replace the given values at the boundary nodes and, simultaneously, introduce unknown Lagrangian multipliers, which actually account for the reaction of the boundary constraints.

Having obtained the solution of the system (4.5), we can proceed to the second stage, that is the solution of the canonical momentum equation. Say $\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^N$ be the solution of (4.5). Taking these points as the nodes of a discretiaztion in the deformed configuration B, we write

$$\mathsf{B} = \bigcup_{e=1}^{N_{el}} \omega_e. \tag{4.7}$$

Let us denote now with M_e^I the shape functions for the element e, then the finite element solution and the corresponding variation can be written

$$\mathbf{g}^e(\mathbf{x}) = \sum_{I=1}^{m} \mathbf{X}^I M_e^I(\mathbf{x}), \quad \delta\mathbf{g}^e(\mathbf{x}) = \sum_{I=1}^{m} \delta\mathbf{X}^I N_e^I(\mathbf{x}). \tag{4.8}$$

Consequently, their gradients take the form

$$\nabla \mathbf{g}^e = \sum_{I=1}^{m} \mathbf{X}^I \otimes \nabla N_e^I, \quad \nabla\delta\mathbf{g}^e = \sum_{I=1}^{m} \delta\mathbf{X}^I \otimes \nabla N_e^I. \tag{4.9}$$

The residual at the node I of the element e will be

$$r_{e,g}^I = \int_{\omega_e} \nabla M_e^I(\mathbf{x}) \cdot (\mathbf{C} + \mathbf{G}^{-1}\tilde{V})dx, \tag{4.10}$$

which provides the following non-linear system of equations for the global nodes

$$R_g^I(\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^N) = \mathop{\mathrm{A}}_{e=1}^{n_{el}} r_{e,g}^I = 0, \quad I = 1, \ldots, N. \tag{4.11}$$

The above equation represents the equilibrium of the material forces at the global node $\mathbf{x}^I$. Actually, it is the discrete version of eq. (2.19). Its nonlinearity stems from the fact that the Eshelby–like tensor $\mathbf{C}$ is related to $\nabla\mathbf{g}$ through a non–linear relation, i.e. eq. (2.15). One can solve the system (4.11) by the use of the Newton–Raphson method. The boundary conditions are of Dirichlet type as they were in the previous problem. Lagrangian multipliers are now necessarily used and this time they interpret the "material reaction" of the boundary constraints, i.e., real material forces at the boundary nodes. It is useful to take the values $\mathbf{X}_0^1, \mathbf{X}_0^2, \ldots, \mathbf{X}_0^N$ as initial guess, because by this way we obtain to compute the discrete material forces corresponding to the approximate deformation $\mathbf{x} = \mathbf{f}^h(\mathbf{X})$ at the interior nodes as well as the real material forces at the boundary nodes. Indeed, the material forces

$$R_g^I(\mathbf{X}_0^1, \mathbf{X}_0^2, \ldots, \mathbf{X}_0^N), \quad I = 1, \ldots, N. \tag{4.12}$$

are nothing but the discrete form of eq. (3.1). The material forces computed at the boundary nodes are expected to be much larger in comparison with the corresponding ones at the interior nodes because, apart from the numerical error, the former represent real material forces that cannot be made to vanish. The solution $\mathbf{X}_1^1, \mathbf{X}_1^2, \ldots, \mathbf{X}_1^N$ of the system (4.11) will provide vanishing material forces at the interior nodes for the specific positions $\mathbf{x}_0^1, \mathbf{x}_0^2, \ldots, \mathbf{x}_0^N$ in the deformed configuration. Thus, it will be a better discretization of B_R. This allows one to solve the equilibrium equation again, with the new discretization, to obtain a better approximate solution.

5 A computational example

Consider a homogeneous, one–dimensional bar clamped on both sides and loaded by a constant load b_0 as it is shown in the Figure 1. The density of the total potential energy is given by

$$U(x, x') = W(x') + V(x) = W(x') - b_0 x, \tag{5.1}$$

where $x = x(X)$, $X \in [0, L]$ is the unknown deformation function. Also, we assume that the stored energy density is a quadratic function of the deformation gradient:

$$W(x') = \frac{1}{2}Ex'^2, \tag{5.2}$$

where E is the elasticity constant.

The total potential energy of the body can be computed by integration of the density U over the entire bar length e.g.

$$\mathbf{I}[x] = \int_0^L U(x, x')dX. \tag{5.3}$$

The equilibrium equation corresponding to the above energy functional, for a bar of unit length, is given by

$$Ex'' + b_0 = 0, \tag{5.4}$$

with boundary conditions

$$x(0) = 0 \qquad \text{and} \qquad x(1) = 1. \tag{5.5}$$

One can easily find the exact solution for the boundary value problem (5.4)–(5.5) which is of the form

$$x(X) = -\frac{b_0}{2E}X^2 + (1 + \frac{b_0}{2E})X. \tag{5.6}$$

We give now the total potential energy functional in terms of the inverse deformation function $X = X(x)$

$$\tilde{\mathbf{I}}[X] = \int_0^L \tilde{U}(x, X')dx, \tag{5.7}$$

where

$$\tilde{U}(x, X') = \tilde{W}(X') + \tilde{V}(x, X'), \tag{5.8}$$

and

$$\tilde{W}(X') = \frac{E}{2}\frac{1}{X'}, \qquad \tilde{V}(x, X') = -b_0 X' x. \tag{5.9}$$

Notice that the external potential energy density depends on both the spatial position x and the gradient of the material position X'. The corresponding equation for the equilibrium of the material forces (material momentum equation) takes the form

$$\frac{X''}{X'^3}E + b_0 = 0, \tag{5.10}$$

with the boundary conditions

$$X\,(0) = 0 \qquad \text{and} \qquad X\,(1) = 1. \tag{5.11}$$

Next, we present some numerical results concerning the above problem. To this end, the analysis of the preceding section has been applied by the use of linear shape functions. We examine three cases with regard to the number and the position of the node points.

Case 1: *A mesh of three nodes.*
We start with an example in which the unique free node X_2 is set in a position far from the optimum. As illustrated in Figure 1a1, the single node moves from the initial position, i.e., $X_2 = 0.9$ to the optimum one, at the middle of the bar. The proposed scheme needs 8 iterations to obtain the value 0.5000. Also, the exact solution for the corresponding displacement field is drawn with red solid line in Fig.

1b1. In the same figure, the FE solution corresponding to the initial mesh (green dash line) and the FE solution for the optimum mesh (blue solid line) are drawn for comparison. The behavior of the total potential energy versus the number of iterations is depicted in Figure 1c1. We remind that the optimum mesh is one whereby the total potential energy of the problem becomes minimum. It is clearly shown in Figure 1c1, that the system obtains lower energy at each iteration as well as that the lowest possible value of the total potential energy corresponds to the optimum mesh ($X_2 = 0.5000$).

Case 2: *A mesh of four nodes.*
In this case we choose an initial mesh which consists of 4 nodes, that is two fixed boundary nodes and two interior nodes free to move. The optimum mesh corresponds to a uniform distribution of nodes due to the symmetry of the solution (see the exact solution in Fig. 1b2). The choice of the initial mesh is arbitrary and corresponds to the values $X_2 = 0,2$ and $X_3 = 0,7$. This mesh is depicted in Fig. 1a2, where the non-vanishing material forces at the interior nodes are sketched by green arrows. The values of the material forces at the boundary nodes - not drawn in the figure - are 3×10^2 times bigger than those ones at the interior nodes. More precisely, we take the following values for the discrete material forces

$$\mathbf{B}_1^c \simeq 3.6 \times 10^2, \quad \mathbf{B}_2^c \simeq -1.6, \quad \mathbf{B}_3^c \simeq 1.2, \quad \mathbf{B}_4^c \simeq -3.7 \times 10^2.$$

After 25 iterations, we obtain the values $X_2 = 0,33333$ and $X_3 = 0,66666$. There, the material forces at the boundary nodes retain the same order of magnitude ($\simeq \pm 3.7 \times 10^2$), while the material forces at the interior nodes essentially vanish ($\simeq 10^{-6}$). Hence, the optimum position of the nodes has been obtained. This is verified also by the behavior of the total potential energy which moves asymptotically to its minimum as shown in Fig. 1c2. It is noted that the value of the total potential energy for the exact solution is $\mathbf{I}_{exact} = -10.10$. The FE solution for displacement field for the initial as well as for the optimum mesh are shown with green dash and blue solid lines, respectively in Fig. 1b2. For the sake of comparison, the exact solution of the displacement field is also drawn there with red solid line.

Case 3: *A rough initial mesh*
Finally, to verify the ability of the proposed scheme to recover the optimum mesh under irregular conditions, we present an example with a rough initial mesh which consists of 4 nodes. The two free to move nodes are set almost one over the other at the positions $X_2 = 0.1$, $X_3 = 0.11$, respectively (see Fig. 2a). As in the Case 2, it took 25 iterations to catch the optimum uniform mesh $X_2 = 0,33333$ and $X_3 = 0,66666$, where the material forces practically vanish (Fig. 2b). The FE solution for displacement field corresponding to the rough mesh is illustrated in Fig. 3a with green dash line. Notice that due to the closeness of the two internal nodes, it seems that it is a three node FE solution. The behavior of the total potential energy versus the number of iterations is depicted in Fig. 3b as well.

6 Conclusions

We have presented a theoretical framework for the formulation of the canonical momentum equation in weak and strong form in order to apply it to the finite element method. To this end, we have used an extremum principle for the total potential energy which was expressed in terms of the inverse deformation function. Besides the standard formulation of the FEM for the Dirichlet boundary problem of elastostasics, we have derived the FE formulation for the Dirichlet problem of the canonical momentum equation. The two problems, being equivalent in the continuous case, supplement each other in the finite dimensional case. The FE solution of the former provides an optimum deformation while the corresponding FE solution of the latter provides an optimum distribution for the node points of the mesh. We have shown that the two problems can be linked in the sense that the numerical solution of one provides a good discretization for the other. By this way, we have established an iterative scheme whereby at every step it is made sure that the energy of the system decreases.

A computational example from one dimensional elastostatics was used to confirm our theoretical predictions. All the numerical examples, that we have tried, show the reduction of the material forces as well as that of the total potential energy. However, we feel that more computational experiments are needed to test the scheme in the two dimensional case. Also, the formulation of a corresponding scheme for the Neumann boundary problem is among our projects for the near future.

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References

[1] Balassas, K.G., Kalpakides, V.K. and Stavroulakis, G.E. 2004 On the use of material forces in the finite element method, In: *Configurational Mechanics*, pp. 157-166, Eds. V.K. Kalpakides and G.A. Maugin, Balkema, Leiden.

[2] Belytschko, T., Liu, W.K. and Moran, B. 2000 *Nonlinear Finite Elements for Continua and Structures*, Wiley.

[3] Braun, M. 1997 Configurational forces induced by finite-element discretization, *Proc. Estonian Acad. Sci. Phys. Math.* **46**, 24-31.

[4] Eshelby, J.D. 1975 The elastic energy-momentum tensor, *J. Elasticity*, **5**, 321-326.

[5] Gurtin, M.E. 2000 *Configurational Forces as Basic Concepts of Continuum Physics*, Applied Mathematical Sciences, **37**, New York, Springer.

[6] Kuhl, E. and Steinmann, P. 2004 On the impact of configurational mechanics on computational mechanics, In: *Configurational Mechanics*, pp. 15-29, Eds. V.K. Kalpakides and G.A. Maugin, Balkema, Leiden.

[7] Maugin, G.A. 1993 *Material Inhomogeneities in Elasticity*, London, Chapman and Hall.

[8] Maugin, G.A. 1995 Material forces: Concepts and applications, *Appl. Mech. Rev.*, **48**, 213-245.

[9] Maugin, G.A. 2000 Geometry of material space: Its consequences in modern computational means, *Tech. Mech.*, **20**, 95-104.

[10] Maugin, G.A. and Kalpakides, V.K. 2002 A Hamiltonian formulation of elasticity and thermoelasticity, *J. Phys. A: Math. Gen.*, **35**, 10775-10788.

[11] Mueller, R., Kolling, S. and Gross D. 2002 On configurational forces in the context of the finite element method, *Int. J. for Num. Meth. in Engineering*, **53**, 1557-1574.

[12] Mueller, R. and Maugin, G.A. 2002 On material forces and finite element discretization, *Comp. Mech.*, **29**, 52-60.

[13] Mueller, R., Gross D. and Maugin, G.A. 2004 Use of material forces in adaptive finite element methods, *Comp. Mech.*, (in press).

[14] Podio-Guidugli, P. 2001 Configurational balances via variational arguments, *Interfaces and Free Boundaries*, **3**, 2001, 223-232.

[15] Steinmann, P. 2001 Applications of material forces to hyperelastic fracture mechanics. Part I. Continuum mechanical setting, *Int. J. Solids Struct.*, **37**, 7371-7391.

[16] Steinmann, P. 2002 On spatial and material settings of hyperelastostatic crystal defects, *J. Mech. Phys. Solids*, **50**, 1743-1766.

[17] Steinmann, P. 2002 On spatial and material settings of thermo-hyperelastodynamics, *J. Elasticity*, **66**, 109-157.

[18] Steinmann, P., Ackermann, D. and Barth, F.J. 2001 Applications of material forces to hyperelastic fracture mechanics. Part II. Computational setting, *Int. J. Solids Struct.*, **38**, 5509-5526.

[19] Thoutireddy, P. 2003 *Variational Arbitrary Lagrangian-Eulerian Method*, Ph.D. Thesis, California Institute of Technology, Pasadena, California.

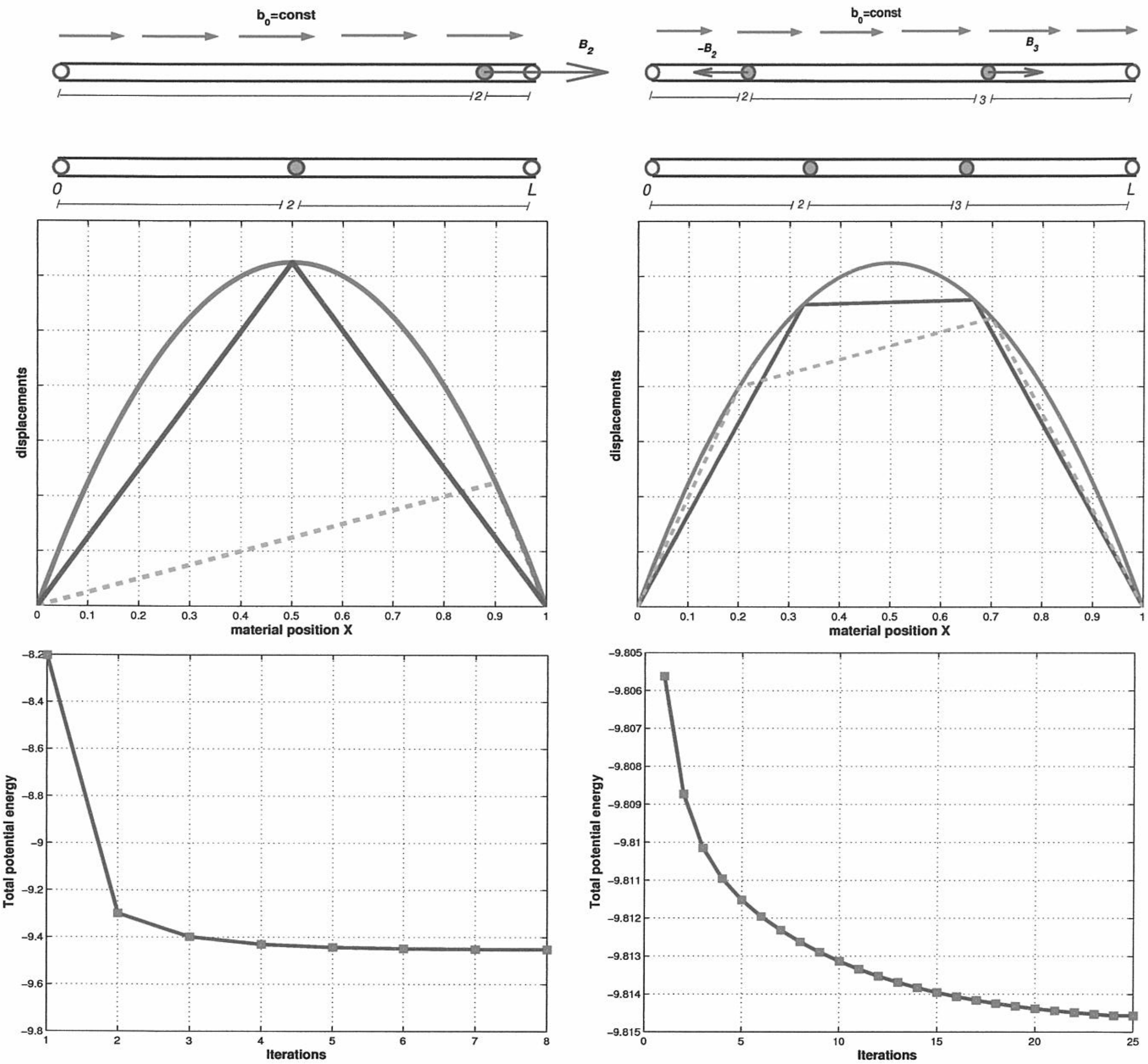

Figure 1: (a1) One-dimensional bar with three nodes. Arbitrary initial mesh-Optimum mesh. (a2) One-dimensional bar with four nodes. Arbitrary initial mesh-Optimum mesh. (b1) The displacement field for the mesh of three nodes. (b2) The displacement field for the mesh of four nodes. (c1) Behaviour of the total potential energy during iterations for three nodes. (c2) Behaviour of the total potential energy during iterations for four nodes.

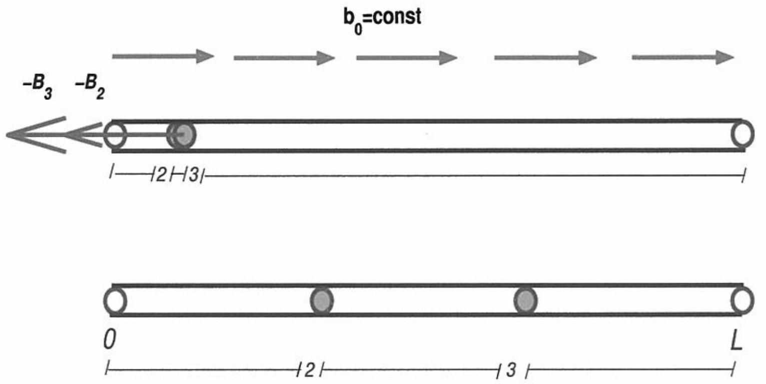

Figure 2: One-dimensional bar with four nodes.(a) Rough initial mesh. (b) Optimum mesh.

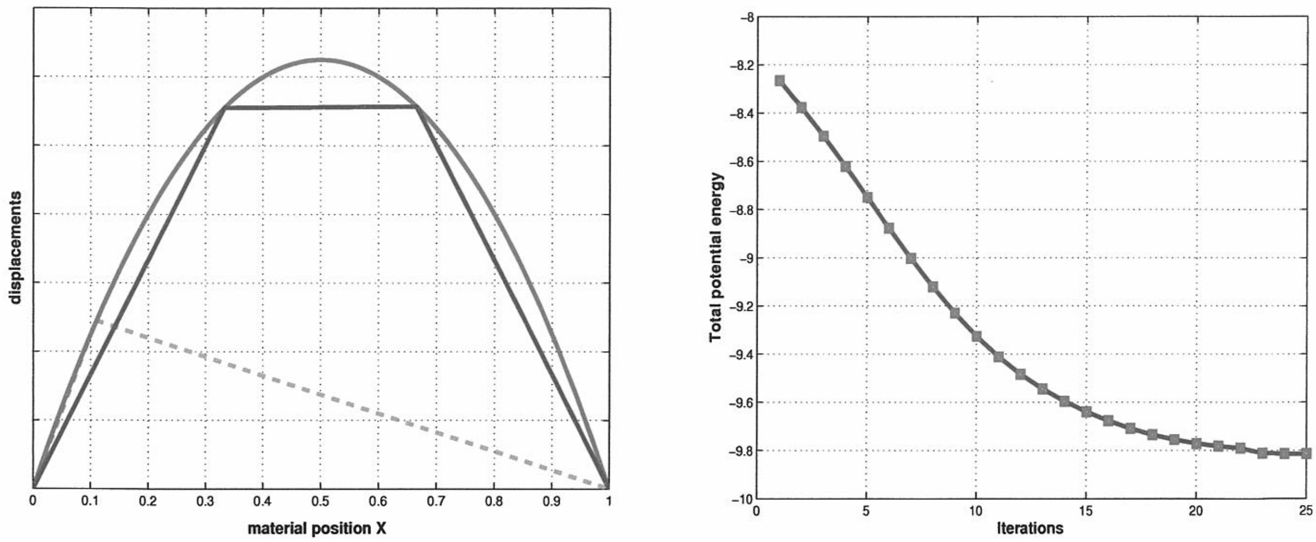

Figure 3: (a) The displacement field. (b) Change of total potential energy during iterations.

Oscillation Criteria for First Order Delay Difference Equations

I.P. STAVROULAKIS

Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
E-mail: ipstav@cc.uoi.gr

Abstract. Oscillation criteria for all solutions of the first order delay difference equation of the form

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ...,$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer are established especially in the case that the well-known oscillation conditions

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

are not satisfied.

Key words: Oscillation, nonoscillation, delay difference equation.

AMS Subject Classification (2000): 39A 11.

1. INTRODUCTION

In the last few decades the oscillation theory of delay differential equations has been extensively developed. The oscillation theory of discrete analogues of delay differential equations has also attracted growing attention in the recent few years. The reader is referred to [1-16, 18-32] and the references cited therein. In particular, the problem of establishing sufficient conditions for the oscillation of all solutions of the delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ..., \tag{1.1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer, has been the subject of many recent investigations. See, for example, [2-9, 12-16, 18-27, 29-32] and the references cited therein. Strong interest in Eq. (1.1) is motivated by the fact that it represents a discrete analogue of the delay differential equation (see [17] and the references cited therein)

$$x'(t) + p(t)x(t-\tau) = 0, \quad p(t) \geq 0, \quad \tau > 0. \tag{1.2}$$

By a solution of (1.1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (1.1) for $n \geq 0$. A solution $\{x_n\}$ of (1.1) is said to be *oscillatory* if the terms x_n of the solution are not eventually positive or eventually negative. Otherwise the solution is called *nonoscillatory*.

For convenience, we will assume that inequalities about values of sequences are satisfied eventually for all large n.

In this paper, our main purpose is to derive new oscillation conditions for all solutions to Eq. (1.1), especially in the case that the known oscillation conditions (see below)

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \text{ and } \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

are not satisfied.

2. OSCILLATION CRITERIA FOR EQ. (1.1)

In 1981, Domshlak [3] was the first who studied this problem in the case where $k = 1$. Then, in 1989 Erbe and Zhang [9] established the following oscillation criteria for Eq. (1.1).

Theorem 2.1.([9]) *Assume that*

$$\beta := \liminf_{n\to\infty} p_n > 0 \ \textit{and} \ \limsup_{n\to\infty} p_n > 1-\beta \tag{C_1}$$

Then all solutions of Eq. (1.1) *oscillate.*

Theorem 2.2.([9]) *Assume that*

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}} \tag{C_2}$$

Then all solutions of Eq. (1.1) *oscillate.*

Theorem 2.3.([9]) *Assume that*

$$A := \limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \tag{C_3}$$

Then all solutions of (1.1) *oscillate.*

In the same year 1989 Ladas, Philos and Sficas [13] proved the following theorem.

Theorem 2.4.([13]) *Assume that*

$$\liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}. \tag{C_4}$$

Then all solutions of Eq. (1.1) *oscillate.*

Therefore they improved the condition (C_2) by replacing the p_n of (C_2) by the arithmetic mean of the terms $p_{n-k}, \ldots, p_{n-1}$ in (C_4).

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (C_2) and (C_4) it should be empasized that, as it is shown in [9], if

$$\sup p_n < \frac{k^k}{(k+1)^{k+1}} \tag{N_1}$$

then Eq. (1.1) has a nonoscillatory solution.

In 1990, Ladas [12] conjectured that Eq. (1.1) has a nonoscillatory solution if

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \leq \frac{k^k}{(k+1)^{k+1}}$$

holds eventually. However this conjecture is not true and a counterexample was given in 1994 by Yu, Zhang and Wang [30].

It is interesting to establish sufficient conditions for the oscillation of all solutions of (1.1) when (C_3) and (C_4) are not satisfied. (For Eq. (1.2), this question has been investigated by many authors, see, for example, [17] and the references cited therein).

In 1993, Yu, Zhang and Qian [29] and Lalli and Zhang [14], trying to improve (C_3), established the following (false) sufficient oscillation conditions for Eq. (1.1)

$$0 < \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad A > 1 - \frac{\alpha^2}{4} \tag{F_1}$$

and

$$\sum_{i=n-k}^{n} p_i \ge d > 0 \text{ for large } n \text{ and } A > 1 - \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1} \tag{F_2}$$

respectively.

Unfortunately, the above conditions (F_1) and (F_2) are not correct. This is due to the fact that they are based on the following (false) discrete version of Koplatadze-Chanturia Lemma. (See [6] and [2]).

Lemma A (False). *Assume that $\{x_n\}$ is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n} p_i \ge M > 0 \quad \textit{for large } n. \tag{1.3}$$

Then

$$x_n > \frac{M^2}{4} x_{n-k} \quad \textit{for large } n.$$

As one can see, the erroneous proof of Lemma A is based on the following (false) statement. (See [6] and [2]).

Statement A (False). *If (1.3) holds, then for any large N, there exists a positive integer n such that $n - k \le N \le n$ and*

$$\sum_{i=n-k}^{N} p_i \ge \frac{M}{2}, \quad \sum_{i=N}^{n} p_i \ge \frac{M}{2}.$$

It is obvious that all the oscillation results which have made use of the above Lemma A or Statement A are not correct. For details on this problem see the paper by Cheng and Zhang [2].

Here it should be pointed out that the following statement (see [13], [20]) is correct and it should not be confused with the Statement A.

Statement 2.1. ([13], [20]) *If*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \textit{for large} \quad n, \tag{1.4}$$

then for any large n, there exists a positive integer n^* with $n - k \leq n^* \leq n$ such that

$$\sum_{i=n-k}^{n^*} p_i \geq \frac{M}{2}, \quad \sum_{i=n^*}^{n} p_i \geq \frac{M}{2}.$$

In 1995, Stavroulakis [20], based on this correct Statement 2.1, proved the following theorem.

Theorem 2.5.([20]) *Assume that*

$$0 < \alpha \leq \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} p_n > 1 - \frac{\alpha^2}{4}. \tag{C_5}$$

Then all solutions of Eq. (1.1) *oscillate.*

In 1998, Domshlak [5], studied the oscillation of all solutions and the existence of nonoscillatory solution of Eq. (1.1) with r -periodic positive coefficients $\{p_n\}, p_{n+r} = p_n$. It is very important that in the following cases where $\{r = k\}, \{r = k+1\}, \{r = 2\}, \{k = 1, r = 3\}$ and $\{k = 1, r = 4\}$ the results obtained are stated in terms of necessary and sufficient conditions and it is very easy to check them.

Following this historical (and chronological) review we also mention that in the case where

$$\frac{1}{k}\sum_{i=n-k}^{n-1} p_i \geq \frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{k}\sum_{i=n-k}^{n-1} p_i = \frac{k^k}{(k+1)^{k+1}},$$

the oscillation of (1.1) has been studied in 1994 by Domshlak [4] and in 1998 by Tang [21] (see also Tang and Yu [23]). In a case when p_n is asymptotically close to one of the periodic critical states, unimprovable results about oscillation preperties of the equation

$$x_{n+1} - x_n + p_n x_{n-1} = 0$$

were obtained by Domshlak in 1999 [7] and in 2000 [8].

In 1999, Domshlak [6] and in 2000 Cheng and Zhang [2] established the following lemmas, respectively, which may be looked upon as (exact) discrete versions of Koplatadze-Chanturia Lemma.

Lemma 2.1. ([6]) *Assume that $\{x_n\}$ is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \text{for large } n. \tag{1.4}$$

Then

$$x_n > \frac{M^2}{4} x_{n-k} \quad \text{for large } n. \tag{1.5}$$

Lemma 2.2. ([2]) *Assume that $\{x_n\}$ is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \text{for large } n. \tag{1.4}$$

Then

$$x_n > M^k x_{n-k} \quad \text{for large } n. \tag{1.6}$$

Based on these lemmas we establish the following theorem.

Theorem 2.6. *Assume that*

$$0 < \alpha \leq \left(\frac{k}{k+1}\right)^{k+1}.$$

Then either one of the conditions

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{4} \tag{C_6}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \alpha^k \tag{C_7}$$

implies that all solutions of Eq. (1.1) *oscillate.*

Proof. Assume, for the sake of contadiction, that $\{x_n\}$ is an eventually positive solution of Eq. (1.1). Then eventually

$$\Delta x_n = x_{n+1} - x_n \le -p_n x_{n-k} \le 0,$$

and so $\{x_n\}$ is an eventually nonincreasing sequence of positive numbers. Summing up (1.1) from $n-k$ to $n-1$, we have

$$x_n - x_{n-k} + \sum_{i=n-k}^{n-1} p_i x_{i-k} = 0,$$

and, because $\{x_n\}$ is eventually nonincreasing, it follows that for all sufficiently large n

$$x_n - x_{n-k} + \left(\sum_{i=n-k}^{n-1} p_i \right) x_{n-k} \le 0,$$

or

$$x_{n-k} \left(\sum_{i=n-k}^{n-1} p_i + \frac{x_n}{x_{n-k}} - 1 \right) \le 0.$$

Now, using Lemma 2.1, for all sufficiently large n, we have

$$x_{n-k} \left(\sum_{i=n-k}^{n-1} p_i + \frac{\alpha^2}{4} - 1 \right) \le 0$$

or, using Lemma 2.2, for all sufficiently large n, we have

$$x_{n-k} \left(\sum_{i=n-k}^{n-1} p_i + \alpha^k - 1 \right) \le 0$$

respectively, which, in view of (C_6) and (C_7), lead to a contradiction. The proof is complete.

Remark 2.1. From the above theorem it is now clear that

$$0 < \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1} \right)^{k+1} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{4}$$

is the correct oscillation condition by which the (false) condition (F_1) should be replaced.

Remark 2.2. Observe the following:

(i) When $k = 1, 2$

$$\alpha^k > \frac{\alpha^2}{4},$$

(since, from the above mentioned conditions, it makes sense to investigate the case when $\alpha < \left(\frac{k}{k+1}\right)^{k+1}$) and therefore condition (C_6) implies (C_7).

(ii) When $k = 3$,

$$\alpha^3 > \frac{\alpha^2}{4} \text{ when } \alpha > \frac{1}{4}$$

while

$$\alpha^3 < \frac{\alpha^2}{4} \text{ when } \alpha < \frac{1}{4}.$$

So in this case the conditions (C_6) and (C_7) are independent.

(iii) When $k \geq 4$

$$\alpha^k < \frac{\alpha^2}{4},$$

and therefore condition (C_7) implies (C_6).

(iv) When $k < 12$ condition (C_6) or (C_7) implies (C_3).

(v) When $k \geq 12$ condition (C_6) may hold but condition (C_3) may not hold.

We illustrate these by the following examples.

Example 2.1. Consider the equation

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \quad n = 0, 1, 2, ...,$$

where

$$p_{2n} = \frac{1}{10}, \quad p_{2n+1} = \frac{1}{10} + \frac{64}{95} \sin^2 \frac{n\pi}{2}, \qquad n = 0, 1, 2, ...,$$

Here $k = 3$ and it is easy to see that

$$\alpha = \liminf_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} < \left(\frac{3}{4}\right)^4$$

and

$$\limsup_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} + \frac{64}{95} > 1 - \alpha^3.$$

Thus condition (C_7) is satisfied and therefore all solutions oscillate. Observe, however, that condition (C_6) is not satisfied.

If, on the other hand, in the above equation

$$p_{2n} = \frac{8}{100}, \quad p_{2n+1} = \frac{8}{100} + \frac{746}{1000}\sin^2\frac{n\pi}{2}, \qquad n = 0, 1, 2, ...,$$

then it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} < \left(\frac{3}{4}\right)^4$$

and

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} + \frac{746}{1000} > 1 - \frac{\alpha^2}{4}.$$

In this case condition (C_6) is satisfied and therefore all solutions oscillate. Observe, however, that condition (C_7) is not satisfied.

Example 2.2. Consider the equation

$$x_{n+1} - x_n + p_n x_{n-16} = 0, \quad n = 0, 1, 2, ...,$$

where

$$p_{17n} = p_{17n+1} = ... = p_{17n+15} = \frac{2}{100}, \ p_{17n+16} = \frac{2}{100} + \frac{655}{1000}, \quad n = 0, 1, 2, ...$$

Here $k = 16$ and it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-16}^{n-1} p_i = \frac{32}{100} < \left(\frac{16}{17}\right)^{17}$$

and

$$\limsup_{n\to\infty} \sum_{i=n-16}^{n-1} p_i = \frac{32}{100} + \frac{655}{1000} = 0.975 > 1 - \frac{\alpha^2}{4}.$$

We see that condition (C_6) is satisfied and therefore all solutions oscillate. Observe, however, that

$$A = \limsup_{n\to\infty} \sum_{i=n-16}^{n} p_i = \frac{34}{100} + \frac{655}{1000} = 0.995 < 1$$

that is, condition (C_3) is not satisfied.

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REFERENCES

1. R.P. Agarwal and P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer Academic Publishers, 1997.

2. S. S. Cheng and G. Zhang, "Virus" in several discrete oscillation theorems, *Applied Math. Letters*, **13** (2000), 9-13.

3. Y. Domshlak, Discrete version of Sturmian Comparison Theorem for non-symmetric equations, *Doklady Azerb.Acad.Sci.* **37** (1981), 12-15 (in Russian).

4. Y. Domshlak, Sturmian comparison method in oscillation study for discrete difference equations, I, *Differential and Integral Equations*, **7** (1994), 571-582.

5. Y. Domshlak, Delay-difference equations with periodic coefficients: sharp results in oscillation theory, *Math. Inequal. Appl.*, **1** (1998), 403-422.

6. Y. Domshlak, What should be a discrete version of the Chanturia-Koplatadze Lemma? *Funct. Differ. Equ.*, **6** (1999), 299-304.

7. Y. Domshlak, Riccati Difference Equations with almost periodic coefficients in the critical state, *Dynamic Systems Appl.*, **8** (1999), 389-399.

8. Y. Domshlak, The Riccati Difference Equations near "extremal" critical states, *J. Difference Equations Appl.*, **6** (2000), 387-416.

9. L. Erbe and B.G. Zhang, Oscillation of discrete analogues of delay equations, *Differential and Integral Equations*, **2** (1989), 300-309.

10. I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

11. J. Jaroš and I.P. Stavroulakis, Necessary and sufficient conditions for oscillations of difference equations with several delays, *Utilitas Math.*, **45** (1994), 187-195.

12. G. Ladas, Recent developments in the oscillation of delay difference equations, In *International Conference on Differential Equations, Stability and Control*, Dekker, New York, 1990.

13. G. Ladas, C. Philos and Y. Sficas, Sharp conditions for the oscillation of delay difference equations, *J. Appl. Math. Simulation*, **2** (1989), 101-112.

14. B. Lalli and B.G. Zhang, Oscillation of difference equations, *Colloquium Math.*, **65** (1993), 25-32.

15. Zhiguo Luo and J.H. Shen, New results for oscillation of delay difference equations, *Comput. Math. Appl.* **41** (2001), 553-561.

16. Zhiguo Luo and J.H. Shen, New oscillation criteria for delay difference equations, *J. Math. Anal. Appl.* **264** (2001), 85-95.

17. Y.G. Sficas and I.P. Stavroulakis, Oscillation criteria for first-order delay differential equations, *Bull. London Math. Soc.* **35** (2003), no.2, 239-246.

18. J.H. Shen and Zhiguo Luo, Some oscillation criteria for difference equations, *Comput. Math. Applic.*, **40** (2000), 713-719.

19. J.H. Shen and I.P. Stavroulakis, Oscillation criteria for delay difference equations, Univ. of Ioannina, T. R. N^0 4, 2000, *Electron. J. Diff. Eqns. Vol.* **2001** (2001), no.10, pp. 1-15.

20. I.P. Stavroulakis, Oscillation of delay difference equations, *Comput. Math. Applic.*, **29** (1995), 83-88.

21. X.H. Tang, Oscillations of delay difference equations with variable coefficients, (Chinese), *J. Central South Univ. of Technology*, **29** (1998), 287-288.

22. X.H. Tang and S.S. Cheng, An oscillation criterion for linear difference equations with oscillating coefficients, J. *Comput. Appl. Math..*, **132** (2001), 319-329.

23. X.H. Tang and J.S. Yu, Oscillation of delay difference equations, *Comput. Math. Applic.*, **37** (1999), 11-20.

24. X.H. Tang and J.S. Yu, A further result on the oscillation of delay difference equations, *Comput. Math. Applic.*, **38** (1999), 229-237.

25. X.H. Tang and J.S. Yu, Oscillations of delay difference equations in a critical state, *Appl. Math. Letters*, **13** (2000), 9-15.

26. X.H. Tang and J.S. Yu, Oscillation of delay difference equations, *Hokkaido Math. J.* **29** (2000), 213-228.

27. X.H. Tang and J.S. Yu, New oscillation criteria for delay difference equations, *Comput. Math. Applic.*, **42** (2001), 1319-1330.

28. Weiping Yan and Jurang Yan, Comparison and oscillation results for delay difference equations with oscillating coefficients, *Internat. J. Math. & Math. Sci.*, **19** (1996), 171-176.

29. J.S. Yu, B.G. Zhang and X.Z. Qian, Oscillations of delay difference equations with oscillating coefficients, *J. Math. Anal. Appl.*, **177** (1993), 432-444.

30. J.S. Yu, B.G. Zhang and Z.C. Wang, Oscillation of delay difference equations, *Appl. Anal.*, **53** (1994), 117-124.

31. B.G. Zhang and Yong Zhou, The semicycles of solutions of delay difference equations, *Comput. Math. Applic.*, **38** (1999), 31-38.

32. B.G. Zhang and Yong Zhou, Comparison theorems and oscillation criteria for difference equations, *J. Math. Anal. Appl.*, **247** (2000), 397-409.

Balance Laws in Dynamic Fracture

Eleni K. Agiasofitou & Vassilios K. Kalpakides
Department of Mathematics, Division of Applied Mathematics and Mechanics,
University of Ioannina, Greece; me00377@cc.uoi.gr, vkalpak@cc.uoi.gr

ABSTRACT: This work aims at the study of the dynamic fracture of an elastic material in the framework of the configurational mechanics. The analysis is based on the global balances for the physical and configurational fields. Thus, the concept of the balance law for an elastic fractured body, in Euclidean and material space, is treated in detail. In the spirit of modern continuum mechanics, a rigorous localization process is proposed. This procedure provides the equations in Euclidean and material space as well as the new contributions for the configurational forces and moments at the crack tip. In addition, it facilitates the derivation of the relationship between the energy release rate (or the rotational release rate) and the configurational force (or the configurational moment). The results are compared with the corresponding ones of fracture mechanics and some new interpretations are discussed.

1 INTRODUCTION

The propagation of a crack of any curvature in a deformable body is a complex phenomenon because apart from the dynamics of the elastic motion, the evolution of the crack must be accounted for, too. The evolution of the crack takes place, not in the physical space, but within the material body, that is, the material space. Thus, we believe that configurational mechanics (Maugin 1993, 1995; Gurtin 2000) should be the appropriate framework in which this problem can be efficiently studied.

To this scope, we start with the global balance laws as it is used to do in any other problem in continuum mechanics. More specifically, we consider an elastic body with a propagating crack in its interior and postulate the balances for all relevant fields, included the configurational ones, for any arbitrary part of the body (Agiasofitou and Kalpakides 2003). In the presence of the crack, this procedure becomes much more complicated because of two reasons. Firstly, the involved fields are not continuous across the crack, even more, they may have a singularity at the crack tip and second, the underlying kinematics is more complicated due to the presence of the separate crack kinematics. In particular, the singularities at the crack tip make necessary to reformulate the transport and divergence theorems, which are indispensable for any localization process.

In literature, such a view can be found in the work of (Steinmann 2000) who presented balance laws in both the physical and material space for elastostatics of a smooth elastic body. Also, reports to equations, which can be considered as balance laws for a fractured body, appeared in (Maugin 1993, 1995; Dascalu and Maugin 1995; Gurtin and Podio-Guidugli 1996; Gurtin 2000; Kienzler and Herrmann 2000) but, to the best of our knowledge, up to now there is not a complete and consistent analysis in the spirit of modern continuum mechanics.

The global view adopted in this paper can shed light on the relationship between the configurational fields at the crack tip and the energy release rates as well as the connection between the first ones and the J and L integrals. For instance, starting from the pseudomomentum and energy equations, one can establish a connection between the energy release rate and the configurational force at the crack tip. This quantity is referred to by Maugin as global material force and it is directly related to the J–integral (Maugin 1993; Dascalu and Maugin 1995). One of our goal in this paper is to explore an analogous relation starting from the material angular momentum and energy equations. In this case, it is expected a connection between the configurational moment at the crack tip and the rotational energy release rate. Such a relation has been provided by (Maugin and Trimarco 1995) for the case where the defect is a disclination line.

Furthermore, (Golebiewska Herrmann and Herrmann 1981) considered the case of a stationary crack which rotates and they computed the rotational energy release rate. Also, (Eischen and Herrmann 1987) tried to connect the conservation (and balance) laws with the energy release rates and the J, L and M integrals. In these works, a straight stationary crack is considered and the rotational energy release rate emerges by a virtual rotation of the crack around its center. Although this is a very successful and meaningful manipulation (in the sense that the associated conservation law is coming from the invariance of the action functional under the group of rotations), it can not be related to a real situation of a propagating crack.

Looking for a more physical interpretation, the propagation of a crack along a curve of arbitrary curvature is considered in such a way that the linear and the angular velocity of the crack tip to be inserted. The balance laws are postulated and from the localization process the configurational fields at the crack tip naturally arise. Finally, these quantities are correlated with the energy release rates and the J and L integrals of fracture mechanics.

Although the crack propagation in a deformable body is a dissipative phenomenon, in this paper no mention is made to the second law of thermodynamics and to the subsequent discussion about constitutive relations.

In Section 2, some preliminaries concerning the proper kinematics for a cracked elastic body are presented. In Section 3, an abstract balance law is postulated, the conditions under which it is meaningful are examined and its consequences are extracted rigorously. The application of this procedure to the physical and configurational fields, related to the problem under study, is made in Sections 4 and 5, respectively. Finally, in Section 6, the obtained results are used to derive the relations between the energy release rates and the configurational fields at the crack tip.

2 PRELIMINARIES

Let $\mathcal{B}_R$ be the reference configuration containing a crack which is described by a smooth, non-intersecting curve C_R with the one end point to lie on the boundary of the body and the other one to be the crack tip, $\mathbf{Z}_0$. We consider that the crack evolves, not necessarily in straight direction, following the "motion" of the crack tip within the body. Thus at the time t, the crack is represented by a smooth curve $C(t)$ belonging to a material configuration $\mathcal{B}_t,\ t \in I \subset I\!R$, where I denotes a time interval. The only difference between the reference configuration $\mathcal{B}_R$ and the material configurations $\mathcal{B}_t$ lies in the different curve they contain. Certainly, it is required for $t_1 > t_2$ to imply $C(t_2) \subset C(t_1)$.

We focus now on the end point of the crack at time t, $\mathbf{Z}(t)$. We consider that $\mathbf{Z}(t)$ is a smooth, time dependent mapping, hence its derivative

$$\mathbf{V}(t) = \frac{d\mathbf{Z}}{dt} \tag{1}$$

provides the propagation velocity of the crack. Also, if we denote with $\mathbf{t}$ the tangent vector to the crack curve, we can write $\mathbf{V} = V\mathbf{t}$.
Taking the standard view of fracture mechanics, we consider a disc of radius ϵ centered at the crack tip $\mathbf{Z}(t)$ for any time t, denoted by $D_\epsilon(t)$:

$$D_\epsilon(t) = \{\mathbf{X} \in \mathcal{B}_t : |\mathbf{X} - \mathbf{Z}(t)| \leq \epsilon\}. \tag{2}$$

At the time t_0, the tip disc is given by:

$$D_{\epsilon 0} = \{\mathbf{Y} \in \mathcal{B}_R : |\mathbf{Y} - \mathbf{Z}_0| \leq \epsilon\}.$$

Notice here that $D_{\epsilon 0} \subset \mathcal{B}_R$ and $D_\epsilon(t) \subset \mathcal{B}_t$. Also, we will denote the part of the crack curve which lies on $D_\epsilon(t)$ with γ_D, i.e., $\gamma_D = D_\epsilon(t) \cap C(t)$.
Taking into account the crack tip evolution, we can establish a fictitious motion of the tip disc (Fig.1) in the following form

$$\mathbf{X} = X(\mathbf{Y}, t), \quad \mathbf{X} \in D_\epsilon(t), \quad \mathbf{Y} \in D_{\epsilon 0},\ t \in I. \tag{3}$$

Without any loss of generality, we assume that this "motion" is a rigid one (Gurtin 1981) and particularly, it is a simple translation which follows the crack tip evolution, that is

$$X(\mathbf{Y}, t) = \mathbf{Y} + \mathbf{Z}(t) - \mathbf{Z}_0, \quad \text{for all } \mathbf{Y} \in D_{\epsilon 0}. \tag{4}$$

It is obvious that every point of $D_{\epsilon 0}$ "moves" with the velocity of the crack tip, i.e.,

$$\mathbf{V}(\mathbf{Y}, t) = \frac{\partial X}{\partial t}(\mathbf{Y}, t) = \frac{d\mathbf{Z}}{dt} = \mathbf{V}(t), \quad \text{for all } \mathbf{Y} \in D_{\epsilon 0}. \tag{5}$$

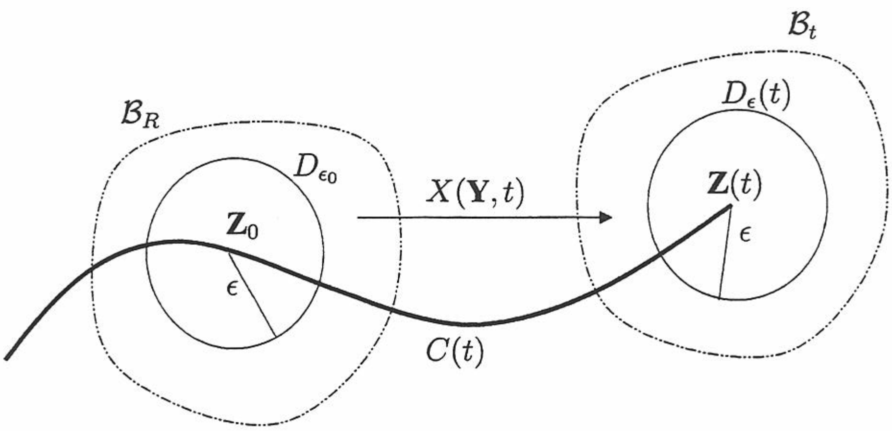

Figure 1: The motion of the tip disc

Consider now the physical motion

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \mathbf{x} \in B_t, \quad \mathbf{X} \in \mathcal{B}_t,\ t \in I \subset I\!R, \tag{6}$$

which is twice-differentiable for all $(\mathbf{X}, t) \in (\mathcal{B}_t \setminus C(t)) \times I$. Also, it is continuous along the crack curve $C(t) \setminus \mathbf{Z}(t)$, as we assume that the crack faces are in perfect contact. We observe that the material points $\mathbf{X} \in D_\epsilon(t)$ depend on t via the mapping X, while the material points $\mathbf{X} \in \mathcal{B}_t \setminus D_\epsilon(t)$ do not depend on t. Consequently, we can compose the mappings X and χ for all $\mathbf{X} \in D_\epsilon(t)$ to interpret both the crack evolution and the motion of the body in the physical space (Fig. 2). Note that this composition holds only for those $\mathbf{X}$ that belong to $D_\epsilon(t)$ at the time t. As a result, we can write for all $\mathbf{X} \in D_\epsilon(t)$

$$\widetilde{\chi} = \chi \circ X, \quad \mathbf{x} = \widetilde{\chi}(\mathbf{Y}, t) = \chi(X(\mathbf{Y}, t), t), \mathbf{Y} \in D_{\epsilon 0}. \tag{7}$$

The partial derivative of $\widetilde{\chi}$ with respect to time will be denoted by $\overset{\circ}{\mathbf{x}}$ and the following chain differentiation will hold

$$\overset{\circ}{\mathbf{x}} = \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}, t)\, \frac{\partial X}{\partial t}(\mathbf{Y}, t) + \frac{\partial \chi}{\partial t}(\mathbf{X}, t), \quad \text{for all} \quad \mathbf{X} \in D_\epsilon(t) \setminus \gamma_D. \tag{8}$$

While for all $\mathbf{X} \in \mathcal{B}_t \setminus D_\epsilon(t)$ away from the crack, it holds

$$\overset{\circ}{\mathbf{x}} = \frac{\partial \chi}{\partial t}(\mathbf{X}, t), \quad \text{for all} \quad \mathbf{X} \in \mathcal{B}_t \setminus D_\epsilon(t).$$

Denoting, as usually, with $\mathbf{F}(\mathbf{X}, t) = \partial \chi(\mathbf{X}, t)/\partial \mathbf{X}$ the deformation gradient and taking into account eq. (5), the equation (8) takes the form

$$\overset{\circ}{\mathbf{x}} = \mathbf{F}(\mathbf{X}, t)\mathbf{V}(t) + \dot{\mathbf{x}}(\mathbf{X}, t) =: \widetilde{\mathbf{V}}(\mathbf{X}, t), \quad \text{for all} \quad \mathbf{X} \in D_\epsilon(t) \setminus \gamma_D, \tag{9}$$

where $\dot{\mathbf{x}}(\mathbf{X}, t) = \partial \chi(\mathbf{X}, t)/\partial t$. Note that, from the above assumptions about the smoothness of x, we have that $\mathbf{F}(\mathbf{X}, t)$ and $\dot{\mathbf{x}}(\mathbf{X}, t)$ are continuous for $\mathbf{X} \in D_\epsilon(t) \setminus \gamma_D$. However, both $\mathbf{F}$ and $\dot{\mathbf{x}}$ are singular at the crack tip $\mathbf{Z}(t)$.

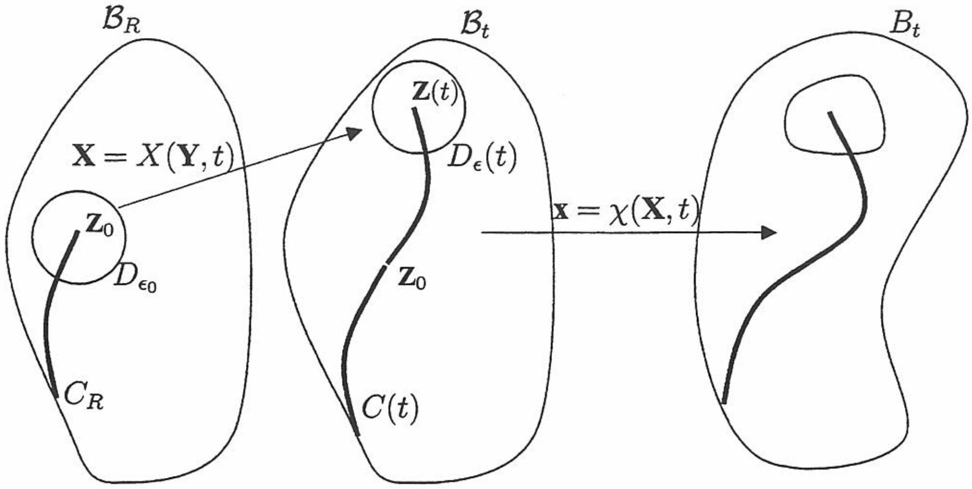

Figure 2: The total motion

The quantity $\widetilde{\mathbf{V}}(\mathbf{X}, t)$ represents the velocity of the deformed tip disc accounting for the crack evolution velocity as well. Though $\widetilde{\mathbf{V}}(\mathbf{X}, t)$ is defined with the aid of the fields $\mathbf{F}$ and $\dot{\mathbf{x}}$ which are singular at the crack tip, we would like $\widetilde{\mathbf{V}}$ to be smooth at the crack tip. Thus, taking the view of (Gurtin 2000, Gurtin and Podio-Guidugli 1996), we assume the existence of a bounded, time–dependent function $\widetilde{\mathbf{U}}(t)$ such that

$$\lim_{\mathbf{X}\to\mathbf{Z}(t)} \widetilde{\mathbf{V}}(\mathbf{X}, t) = \widetilde{\mathbf{U}}(t), \quad \text{uniformly in } I. \tag{10}$$

Notice that the quantity $\tilde{\mathbf{U}}(t)$ represents the velocity of the deformed crack tip.

3 AN ABSTRACT BALANCE LAW FOR A CRACKED BODY

Let Ω be any smooth domain of the body in the material configuration $\mathcal{B}_t$. If the crack tip $\mathbf{Z}(t)$ is an interior point of Ω, then there exists a radius ϵ such that $D_\epsilon(t) \subset \Omega$. In this case, we will denote with Ω_ϵ the subset of Ω which is defined as follows (Fig. 3),

$$\Omega_\epsilon(t) = \Omega \setminus D_\epsilon(t) \;\; \text{or} \;\; \Omega = \Omega_\epsilon(t) \cup D_\epsilon(t). \tag{11}$$

Notice that $\partial\Omega_\epsilon = \partial\Omega \cup \partial D_\epsilon(t)$. Also, the parts of the crack $C(t)$ contained in Ω_ϵ and Ω will be denoted by γ_ϵ and γ_Ω, respectively , that is

$$\gamma_\epsilon = C(t) \cap \Omega_\epsilon(t), \quad \gamma_\Omega = C(t) \cap \Omega.$$

In standard continuum mechanics, one has the freedom to formulate a global balance law either in the reference configuration or in the current configuration. In the proposed framework, there are three distinct configurations (Fig.2). We work on a material configuration $\mathcal{B}_t$, in which all the relevant fields should be defined. Let $\phi(\mathbf{X}, t)$ be a scalar valued function defined in $\mathcal{B}_t$, representing some physical quantity, sufficiently smooth away from the crack tip and up to the crack $C(t)$ from either side, thus we let ϕ to have a singularity at the crack tip and to be discontinuous with finite jump along $C(t) \setminus \{\mathbf{Z}(t)\}$.

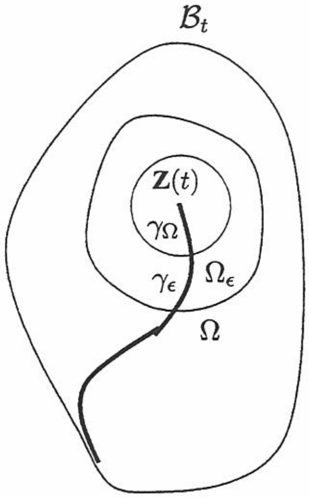

Figure 3: A domain Ω containing the crack tip

Taking the view of (Gurtin 2000), we will assume *the integrability of ϕ in the sense of Cauchy principal value*, i.e. for all $\Omega \in \mathcal{B}_t$

$$\int_{\Omega} \phi(\mathbf{X},t)\, dA = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \phi(\mathbf{X},t)\, dA. \tag{12}$$

Analogously, it holds for the line integral of a vector valued function $\mathbf{g}(\mathbf{X},t)$ along the curve γ_Ω in the sense

$$\int_{\gamma_\Omega} \mathbf{g}(\mathbf{X},t) \cdot \mathbf{n}\, dl = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \mathbf{g}(\mathbf{X},t) \cdot \mathbf{n}\, dl, \tag{13}$$

where $\mathbf{n}$ is the unit normal to $C(t)$. Hereafter, when we refer to the integrability of any function over Ω and γ_Ω, it will be meant in the sense of eqs. (12) and (13).
Next, we consider a *global balance law for the quantity ϕ* of the form

$$\frac{d}{dt} \int_{\Omega} \phi(\mathbf{X},t)\, dA = \int_{\partial\Omega} \mathbf{f}(\mathbf{X},t) \cdot \mathbf{N}\, dS + \int_{\Omega} h(\mathbf{X},t)\, dA + g(t), \tag{14}$$

where $\mathbf{N}$ is the outward unit normal to the boundary $\partial\Omega$ and $\mathbf{f}$ and h, are the flux and the source of ϕ, respectively. The time dependent function g represents the source of ϕ due to the crack evolution.
It is apparent that the integrability of ϕ is not enough to make eq. (14) meaningful. So, we must pose extra smoothness on the integrands. Denoting with $[\mathbf{f}]$ the jump of $\mathbf{f}$ across the crack, we assume the following conditions

C1 $h,\ \phi$ are integrable over Ω.

C2 $\displaystyle \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \frac{\partial \phi}{\partial t}(\mathbf{X},t)\, dA = \int_{\Omega} \frac{\partial \phi}{\partial t}(\mathbf{X},t)\, dA,$ uniformly in I.

C3 $\displaystyle \int_{\partial D_\epsilon} \phi(\mathbf{X},t)(\mathbf{V} \cdot \mathbf{N})\, dS$ converges uniformly in I as $\epsilon \to 0$.

C4 $\mathrm{Div}\,\mathbf{f},\ [\mathbf{f}] \cdot \mathbf{n}$ are integrable over Ω and γ_Ω, respectively.

C5 $\displaystyle \int_{\partial D_\epsilon} \mathbf{f}(\mathbf{X},t) \cdot \mathbf{N}\, dS$ converges to a time dependent function as $\epsilon \to 0$.

One can prove the following statement (Agiasofitou and Kalpakides 2003)
Assume that the Conditions 1, 2 and 3 hold. Then, $\int_\Omega \phi(\mathbf{X},t)\, dA$ is a differentiable function of t. In addition, its derivative will be given by the relation

$$\frac{d}{dt} \int_{\Omega} \phi(\mathbf{X},t)\, dA = \lim_{\epsilon \to 0} \left(\frac{d}{dt} \int_{\Omega_\epsilon} \phi(\mathbf{X},t)\, dA \right). \tag{15}$$

The transport theorem and the divergence theorem for any domain Ω_ϵ, can be written, respectively

$$\frac{d}{dt} \int_{\Omega_\epsilon} \phi(\mathbf{X},t)\, dA = \int_{\Omega_\epsilon} \frac{\partial \phi(\mathbf{X},t)}{\partial t}\, dA - \int_{\partial D_\epsilon} \phi(\mathbf{X},t)(\mathbf{V} \cdot \mathbf{N})\, dS \tag{16}$$

and

$$\int_{\Omega_\epsilon} \text{Div}\, \mathbf{f}(\mathbf{X},t)\, dA = \int_{\partial\Omega} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\, dS - \int_{\partial D_\epsilon} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\, dS + \int_{\gamma_\epsilon} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl. \tag{17}$$

Using the Conditions 1-5 and the eqs. (16)-(17), one can prove the following versions for the transport theorem and divergence theorem appropriate for the problem under study (Agiasofitou and Kalpakides 2003)

$$\frac{d}{dt}\int_{\Omega} \phi(\mathbf{X},t)\, dA = \int_{\Omega} \frac{\partial\phi}{\partial t}(\mathbf{X},t)\, dA - \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \phi(\mathbf{X},t)(\mathbf{V}\cdot\mathbf{N})\, dS \tag{18}$$

and

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{N}\, dS = \int_{\Omega} \text{Div}\, \mathbf{f}\, dA + \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\, dS - \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl. \tag{19}$$

Inserting eqs. (18) and (19) into eq. (14), we obtain

$$\int_{\Omega}\left(\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dA + \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl - \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} (\phi(\mathbf{X},t)(\mathbf{V}\cdot\mathbf{N}) + \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N})\, dS - g(t) = 0, \tag{20}$$

for all Ω containing the crack tip.
We remark that in the case where Ω does not contain the crack tip and any part of the crack, eq. (20) takes the simpler form

$$\int_{\Omega}\left(\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dA = 0. \tag{21}$$

Thus, due to the arbitrariness of Ω, we conclude that

$$\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t) = 0, \quad \text{for all } t\in I,\ \mathbf{X}\in\mathcal{B}_t\setminus C(t). \tag{22}$$

Similarly, we can consider Ω containing a part of the crack apart from the crack tip. In this case, the global balance law (eq. (20)) takes the form

$$\int_{\Omega}\left(\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dA + \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl = 0. \tag{23}$$

However, $\int_\Omega (\partial\phi(\mathbf{X},t)/\partial t - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t))\ dA = 0$, because its integrand is zero almost everywhere due to eq. (22), i.e., it is zero everywhere apart from the crack line γ_Ω, which is a set of measure zero in Ω. Consequently, eq. (23) gives

$$\int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl = 0, \tag{24}$$

for all γ_Ω which do not contain the crack tip. Thus, we obtain

$$[\mathbf{f}(\mathbf{X},t)] \cdot \mathbf{n} = 0, \quad \text{for all } \; t \in I, \; \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}. \tag{25}$$

In the same line of argument, we consider arbitrary Ω which contains the whole crack. In this case, we must use the complete form of eq. (20). Taking into account the results provided by eqs. (22) and (25), we remark that the integrands of the first two terms of eq. (20) vanish almost everywhere in any Ω_ϵ and any γ_ϵ, respectively, thus we can write

$$\int_{\Omega_\epsilon} \left(\frac{\partial \phi(\mathbf{X},t)}{\partial t} - \operatorname{Div} \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t) \right) dA = 0,$$
$$\int_{\gamma_\epsilon} [\mathbf{f}(\mathbf{X},t)] \cdot \mathbf{n} \, dl = 0,$$

for all $\epsilon > 0$.
Thus, recalling the sense of integrability given by eqs. (12) and (13), we conclude that

$$\int_{\Omega} \left(\frac{\partial \phi(\mathbf{X},t)}{\partial t} - \operatorname{Div} \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t) \right) dA = 0, \tag{26}$$

$$\int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)] \cdot \mathbf{n} \, dl = 0, \tag{27}$$

for all Ω and γ_Ω, even they contain the crack tip. Finally, we obtain the localization of the balance law at the crack tip as follows:

$$g(t) = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\phi(\mathbf{X},t)(\mathbf{V} \cdot \mathbf{N}) + \mathbf{f}(\mathbf{X},t) \cdot \mathbf{N}) dS, \quad \text{for all } \; t \in I. \tag{28}$$

To sum up, the requirement that the balance law (14) holds for all $\Omega \in \mathcal{B}_t$ implies the local equations (22), (25) and (28), that is,

$$\begin{aligned} &\frac{\partial \phi}{\partial t} - \operatorname{Div} \mathbf{f} - h = 0, \qquad \text{for all } \; t \in I, \; \mathbf{X} \in \mathcal{B}_t \setminus C(t), \\ &[\mathbf{f}] \cdot \mathbf{n} = 0, \qquad \text{for all } \; t \in I, \; \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \\ &g(t) = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\phi(\mathbf{V} \cdot \mathbf{N}) + \mathbf{f} \cdot \mathbf{N}) dS, \qquad \text{for all } \; t \in I. \end{aligned} \tag{29}$$

4 BALANCE LAWS IN THE PHYSICAL SPACE

Throughout this and the next section, we assume that each field inserted in a global balance law at the position of the abstract functions Φ, $\mathbf{f}$ and h will enjoy the corresponding smoothness specified in the previous section.

4.1 The balances of mass, momentum and angular momentum

We assume that there are no sources of mass, momentum and angular momentum, due to the crack evolution. Thus, we accept that, apart from the energy, the crack evolution does not intervene directly in the balance of the physical fields. Nevertheless, we expect a new relation at the crack tip due to the singularities of the physical fields. We denote with ρ and $\mathbf{T}$ the mass density in the material configuration and the Piola-Kirchhoff stress tensor, respectively. Also, the position vector of $\mathbf{x}$ is denoted with $\mathbf{r} = \mathbf{x} - \mathbf{0}$. As in the standard continuum mechanics, it is postulated that the mass, the momentum and the angular momentum fulfil the following relations

$$\frac{d}{dt}\int_{\Omega} \rho(\mathbf{X}, t)\, dA = 0, \tag{30}$$

$$\frac{d}{dt}\int_{\Omega} \rho\dot{\mathbf{x}}\, dA = \int_{\partial\Omega} \mathbf{TN}\, dS, \tag{31}$$

$$\frac{d}{dt}\int_{\Omega} \mathbf{r} \times \rho\dot{\mathbf{x}}\, dA = \int_{\partial\Omega} (\mathbf{r} \times \mathbf{T})\mathbf{N}\, dS, \tag{32}$$

for every part Ω of $\mathcal{B}_t$ and for every t in some interval I.
The local form of the balances (30-32) outside the crack are extracted from eq. $(29)_1$

$$\frac{\partial \rho(\mathbf{X}, t)}{\partial t} = 0 \Rightarrow \rho = \rho(\mathbf{X}), \tag{33}$$

$$\frac{\partial}{\partial t}(\rho\dot{\mathbf{x}}) - \mathrm{Div}\mathbf{T} = 0, \tag{34}$$

$$\frac{\partial}{\partial t}(\mathbf{r} \times \rho\dot{\mathbf{x}}) - \mathrm{Div}(\mathbf{r} \times \mathbf{T}) = 0, \tag{35}$$

for all $t \in I,\ \mathbf{X} \in \mathcal{B}_t \setminus C(t)$.
Moreover, the localization process gives the following jump conditions (see eq. $(29)_2$) along the crack curve

$$[\mathbf{T}]\mathbf{n} = 0, \tag{36}$$

$$[\mathbf{r} \times \mathbf{T}]\mathbf{n} = 0, \tag{37}$$

for all $t \in I,\ \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}$. The above local equations and jump conditions do not differ from the corresponding ones holding for any smooth elastic body with a material surface of discontinuity within it. Recalling that the motion $\chi(\mathbf{X}, t)$ is continuous along the crack $C(t)$ (hence, $\mathbf{r}$ is continuous as well), we easily conclude that the condition (37) follows from the jump condition (36).
The new results of the proposed approach concern the relations holding at the crack tip are derived from$(29)_3$ as follows

$$\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \rho(\mathbf{V} \cdot \mathbf{N})\, dS = 0, \tag{38}$$

$$\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\rho\dot{\mathbf{x}}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{TN})\, dS = 0, \tag{39}$$

$$\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \mathbf{r} \times (\rho\dot{\mathbf{x}}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{TN})\, dS = 0, \tag{40}$$

for all $t \in I$.
Equation (38) shows that the rate of mass flow through ∂D_ϵ vanishes, when the boundary shrinks onto the crack tip. Eqs. (39) and (40) represent the balance of linear momentum

and angular momentum at the crack tip, respectively. Adopting the *standard momentum condition* of (Gurtin 2000), that is,

$$\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \rho \dot{\mathbf{x}} \otimes \mathbf{N} \, dS = 0,$$

we take from eq. (39)

$$\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \rho \dot{\mathbf{x}} (\mathbf{V} \cdot \mathbf{N}) \, dS = 0, \quad \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \mathbf{TN} \, dS = 0.$$

4.2 The balance of Energy

Unlike the balances for mass, linear momentum and angular momentum, we assume that the balance of energy is directly influenced by the crack growth. This is quite reasonable because the crack propagation is a dissipative phenomenon, that is to say, the growth of the crack consumes a part of the energy given by the applied forces. Hence, an energy source term describing the total dissipation rate of the body, denoted here by $\Phi(t)$, must be added in the energy balance. Thus, the global balance law for energy can be postulated as

$$\frac{d}{dt} \int_{\Omega} (W + K) \, dA = \int_{\partial \Omega} \mathbf{TN} \cdot \dot{\mathbf{x}} \, dS - \Phi, \quad \text{for all } t \in I, \ \Omega \in \mathcal{B}_t, \tag{41}$$

where W is the elastic energy density and K is the kinetic energy density, both are defined per unit volume in material configuration.

Localizing eq. (41), we obtain (see eq. (29))

$$\frac{\partial}{\partial t}(W + K) - \text{Div}(\mathbf{T}^T \dot{\mathbf{x}}) = 0, \ \ \forall\, t \in I, \ \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{42}$$

$$[\mathbf{T}^T \dot{\mathbf{x}}] \cdot \mathbf{n} = 0, \ \ \forall\, t \in I, \ \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \tag{43}$$

$$\Phi = \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big((W + K)(\mathbf{V} \cdot \mathbf{N}) + \mathbf{T}^T \dot{\mathbf{x}} \cdot \mathbf{N}\big) dS, \ \ \forall\, t \in I. \tag{44}$$

It is obvious that eqs. (42) and (43) are the local energy equation and the associated jump condition along the crack, respectively. Also, eq. (44) is the energy flow out of the body and into the crack tip per unit time and if it be divided by the crack propagation velocity V, it will give the well-known, in fracture literature (Freund 1981), dynamic energy release rate G, i.e.,

$$G = \Phi / V. \tag{45}$$

5 BALANCE LAWS IN THE MATERIAL SPACE

The balances which we are dealt with in the last section do not exhaust all the relevant quantities involved in our problem. We must further consider balances for the configurational fields, that is, the pseudomomentum and the material angular momentum.

5.1 The balance of pseudomomentum

We introduce now the pseudomomentum (or material momentum)

$$\mathcal{P}(\mathbf{X}, t) = -\rho \mathbf{F}^T \dot{\mathbf{x}}, \ t \in I, \ \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{46}$$

a quantity analogous to the physical momentum, concerning changes within the material structure. In a Hamiltonian framework, the pseudomomentum is the dual quantity to the velocity of the inverse motion function, like the physical momentum is the dual of the standard velocity of the body (Maugin and Kalpakides 2002). The contributors to the balance of pseudomomentum will be the material or configurational forces (Maugin 1993). Considering both at a distance and at contact configurational forces, we introduce the configurational body forces $\tilde{\mathbf{f}}$ (source term) and the configurational stress tensor $\mathbf{b}$ (flux term), respectively . Moreover, we consider a pseudomomentum source term, that is a material force, $\mathcal{F} = \mathcal{F}(t)$, produced by the crack evolution. After all these considerations, we postulate the balance law for the pseudomomentum

$$\frac{d}{dt}\int_{\Omega} \mathcal{P}\, dA = \int_{\partial\Omega} \mathbf{bN}\, dS + \int_{\Omega} \tilde{\mathbf{f}}\, dA + \mathcal{F},\ \forall t \in I,\ \forall \Omega \in \mathcal{B}_t. \tag{47}$$

The local equations, obtained by eq. (47), are given (see eqs. (29)) as follows

$$\frac{\partial \mathcal{P}}{\partial t} - \mathrm{Div}\mathbf{b} - \tilde{\mathbf{f}} = 0,\ \ \forall\, t \in I,\ \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{48}$$

$$[\mathbf{b}]\mathbf{n} = 0,\ \ \forall\, t \in I,\ \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \tag{49}$$

$$\mathcal{F} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\mathcal{P}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{bN})\, dS,\ \forall\, t \in I. \tag{50}$$

Eq. (48) is the equation of pseudomomentum, which holds in the smooth part of the body and eq. (49) is the associated jump condition. In addition, eq. (50) represents the material force at the crack tip, which drives the crack evolution. Thus, the quantity $\mathcal{F}$ should be directly related to the energy release rate, G. Also, in the static case it holds the following relation

$$\mathcal{F} = -\lim_{\epsilon \to 0} \mathbf{J}(\epsilon),$$

where $\mathbf{J}$ is the well–known $\mathbf{J}$ -integral of Rice (Budiansky and Rice 1973).
In the absence of a crack or any other rearrangement in the material configuration, the last term in the pseudomomentum balance law vanishes and eq. (48) holds all over the body as a simple identity for the solution of the standard elastic problem. In other words, eqs. (48–50) do not make sense in the standard continuum mechanics, where only the motion in physical space is considered. Thus, the balance law (47) must be considered when one studies any kind of evolution of structural defects. From this point of view, it is a configurational balance law.

Remark 1: One can enrich the balance law (47) by considering an additional term of the form $\int_{\gamma_\Omega} \mathbf{g}^l\, dl$, accounting for configurational forces acting along the crack curve (Gurtin 2000). In that case, the localization process provides

$$[\mathbf{b}]\mathbf{n} + \mathbf{g}^l = 0,$$

instead of eq. (49).

Remark 2: The flux term $\mathbf{b}$, like the Piola–Kirchhoff stress tensor in standard continuum mechanics, needs a constitutive relation to be further determined. Because the constitutive relations are out of the scope of the present procedure, we adopt without reasoning the relation

$$\mathbf{b} = (W - \frac{1}{2}\rho\dot{\mathbf{x}}^2)\mathbf{I} - \mathbf{F}^T\mathbf{T}, \tag{51}$$

that is, the Eshelby stress tensor for the dynamical case (Eshelby 1995). For the derivation and a discussion about this relationship, viewed as a constitutive relation, we refer to the works of (Gurtin 2000) and (Podio-Guidugli 2002). As concerns the term $\tilde{\mathbf{f}}$, we consider it as a distributed body material force, produced by the material inhomogeneities (Maugin 1993). Moreover, the pseudomomentum source term $\mathcal{F}$, produced by the crack evolution, is referred to by (Maugin 1993; Dascalu and Maugin 1995) as *global material force* and by (Gurtin 2000) as *tip traction*.

5.2 The balance of material angular momentum

We proceed to the balance of the material angular momentum, that is, the moment of pseudomomentum, $\mathbf{R}\times\mathcal{P}$, where $\mathbf{R}=\mathbf{X}-\mathbf{0}$ is the position vector of $\mathbf{X}$. The rest contributors to this law should be the moment of material contact and material body forces. Moreover, we consider a material angular momentum source term, $\mathcal{M}=\mathcal{M}(t)$ due to the presence of the crack. We postulate:

$$\frac{d}{dt}\int_{\Omega}\mathbf{R}\times\mathcal{P}\,dA=\int_{\partial\Omega}\mathbf{R}\times\mathbf{bN}\,dS+\int_{\Omega}\mathbf{R}\times\tilde{\mathbf{f}}\,dA+\int_{\Omega}\mathbf{g}\,dA+\mathcal{M},\quad \forall t\in I,\ \forall\Omega\in\mathcal{B}_t, \tag{52}$$

where $\mathbf{g}(\mathbf{X},t)$ is a vector field describing the distribution of material couples within the body.
The localization of eq. (52) provides

$$\frac{\partial(\mathbf{R}\times\mathcal{P})}{\partial t}-\mathrm{Div}(\mathbf{R}\times\mathbf{b})-\mathbf{R}\times\tilde{\mathbf{f}}-\mathbf{g}=0,\ \ \forall t\in I,\ \mathbf{X}\in\mathcal{B}_t\setminus C(t), \tag{53}$$

$$\mathcal{M}=-\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\big(\mathbf{R}\times(\mathcal{P}(\mathbf{V}\cdot\mathbf{N})+\mathbf{bN})\big)\,dS,\ \ \forall t\in I \tag{54}$$

and the associated jump condition

$$[\mathbf{R}\times\mathbf{b}]\mathbf{n}=0,\quad \forall\, t\in I,\ \forall\,\mathbf{X}\in C(t)\setminus\{\mathbf{Z}(t)\},$$

which holds identically due to the continuity of $\mathbf{X}$ and the jump condition (49).
Eq. (53) is the equation of material angular momentum, which holds in the bulk of the body. If $\tilde{\mathbf{f}}=\mathbf{0}$ and $\mathbf{g}=\mathbf{0}$, then it coincides with the corresponding one of (Golebiewska Herrmann 1982).
If we take into account the equation of pseudomomentum (48), then eq. (53) gives the following relation

$$\mathbf{g}=\mathrm{axl}\mathbf{b}\ \text{ or }\ g_A=-e_{ABC}b_{BC}, \tag{55}$$

where axl$\mathbf{b}$ denotes the axial vector of $\mathbf{b}$ (Chadwick 1976). Finally, eq. (53) is written as follows

$$\frac{\partial(\mathbf{R}\times\mathcal{P})}{\partial t}-\mathrm{Div}(\mathbf{R}\times\mathbf{b})-\mathbf{R}\times\tilde{\mathbf{f}}-2\mathrm{axl}\mathbf{b}=0,\ \ \forall t\in I,\ \mathbf{X}\in\mathcal{B}_t\setminus C(t), \tag{56}$$

which is in accordance with the corresponding one of (Steinmann 2000) for the static case.

Remark 3: If the material is homogeneous and isotropic, then the Eshelby stress tensor $\mathbf{b}$ is symmetric (Steinmann 2000; Kalpakides and Agiasofitou 2002), which means that $\mathrm{axl}\mathbf{b} = \mathbf{0}$, so eq. (55) gives $\mathbf{g} = \mathbf{0}$.

Remark 4: Adopting the existence of configurational forces distributed along the crack curve as we did in Remark 1, we obtain the jump condition

$$[\mathbf{R} \times \mathbf{b}]\mathbf{n} + \mathbf{R} \times \mathbf{g}^l = 0.$$

Furthermore, equation (54) gives the form of the *configurational moment $\mathcal{M}$ at the crack tip*. Particularly,

$$\mathcal{M}(t) = -\lim_{\epsilon \to 0} \mathcal{M}_\epsilon(t), \quad \forall t \in I,$$

where

$$\mathcal{M}_\epsilon(t) = \int_{\partial D_\epsilon} \big(\mathbf{R} \times (\mathcal{P}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{b}\mathbf{N})\big)\, dS. \tag{57}$$

The physical interpretation of $\mathcal{M}_\epsilon$ and its possible connection with the L–integral will be examined in the next subsection.

5.3 The configurational moment and the L–integral

It is worth noting that the Eshelby stress tensor used in fracture mechanics literature differs from the one used here. In fracture mechanics, the tensor $\mathbf{b}$ is defined with the aid of the displacement field $\mathbf{u}(\mathbf{X}, t)$, whereas in our analysis it is defined with the aid of the motion mapping $\mathbf{x}(\mathbf{X}, t)$ (see the relation (51)). If we introduce in eq. (51) the displacement field $\mathbf{u}$, we take

$$\mathbf{b} = (W - \frac{1}{2}\rho\dot{\mathbf{x}}^2)\mathbf{I} - \mathbf{F}^T\mathbf{T} = (W - \frac{1}{2}\rho\dot{\mathbf{u}}^2)\mathbf{I} - (\nabla\mathbf{u})^T\mathbf{T} - \tilde{\mathbf{I}}^T\mathbf{T},$$

where $\tilde{\mathbf{I}}$ denotes the two point unit tensor (or the shifter δ_{iA} in a coordinate system). Then, we can write

$$\mathbf{b} = \mathbf{b}^u - \tilde{\mathbf{I}}^T\mathbf{T}, \tag{58}$$

where

$$\mathbf{b}^u = (W - \frac{1}{2}\rho\dot{\mathbf{u}}^2)\mathbf{I} - (\nabla\mathbf{u})^T\mathbf{T}.$$

Notice that using $\mathbf{b}^u$ instead of $\mathbf{b}$ in pseudomomentum equation and neglecting the configurational body forces $\tilde{\mathbf{f}}$, we obtain, in virtue of eq. (34), an equation of the same form

$$\frac{\partial \mathcal{P}^u}{\partial t} - \mathrm{Div}\mathbf{b}^u = 0,$$

where

$$\mathcal{P}^u = -\rho(\nabla\mathbf{u})^T\dot{\mathbf{u}}. \tag{59}$$

However, under the same manipulation the material angular momentum equation does not retain its form. Indeed, inserting eqs. (58) and (59) into eq. (53), neglecting $\tilde{\mathbf{f}}$, $\mathbf{g}$ and taking into account the equation of angular momentum, i.e. eq. (35), we obtain

$$\frac{\partial}{\partial t}(\mathbf{R} \times \mathcal{P}^u + \mathbf{u} \times \rho\dot{\mathbf{u}}) - \mathrm{Div}\,(\mathbf{R} \times \mathbf{b}^u + \mathbf{u} \times \mathbf{T}) = 0. \tag{60}$$

Consequently, eq. (60) must be used in any comparison of the present results with the corresponding ones in the linear fracture mechanics. Indeed, if $\mathbf{R}$ and $\mathbf{T}$ are replaced by the spatial coordinates $\mathbf{x}$ and the Caushy stress tensor $\boldsymbol{\sigma}$, respectively and ρ and W are defined per unit deformed volume, then eq. (60) coincides with the corresponding one of (Fletcher 1975), for a linear, homogeneous and isotropic elastic body in the absence of body forces.
In addition, doing the same replacements in the integral given by eq. (57), the latter becomes

$$\mathcal{M}_\epsilon(t) = \int_{\partial D_\epsilon} \big((\mathbf{R} \times \mathcal{P}^u + \mathbf{u} \times \rho\dot{\mathbf{u}})(\mathbf{V} \cdot \mathbf{N}) + (\mathbf{R} \times \mathbf{b}^u + \mathbf{u} \times \mathbf{T})\,\mathbf{N}\big)\, dS \tag{61}$$

In the static case, apart from the contour of integration, the integral $\mathcal{M}_\epsilon$ reduces to the L–integral, as it was given by (Knowles and Sternberg 1972) and (Steinmann 2000) for a nonlinear, homogeneous and isotropic elastic material. Therefore, an integral having the same integrand with $\mathcal{M}_\epsilon$ along an integration path encircling the total crack can be considered as a generalization of L–integral in the dynamical, non–linear case. It is important to remark that, in fracture mechanics literature, the path of the L–integral includes the whole crack, while in our analysis the path ∂D_ϵ is limited around the crack tip. This, on the one hand, justifies the term "configurational moment at the crack tip" and on the other, provides possibly an alternative physical interpretation of $\mathcal{M}$ and $\mathcal{M}_\epsilon$. More specifically, one can conjecture that the quantity $\mathcal{M}$ is related to the tendency of the crack tip (and as a result of the crack) to turn, while the usual interpretation (for instance, see (Golebiewska Herrmann and Herrmann 1981)) of the L–integral concerns the tendency of a stationary straight crack to rotate, as a whole, with respect to its center.

6 THE ENERGY RELEASE RATES AND THE CONFIGURATIONAL FIELDS

In this section, expressions for the energy release rates will be derived. Particularly, the relationship between the dynamical energy release rate with the configurational force at the crack tip as well as the relationship of the rotational energy release rate with the configurational moment at the crack tip, will be established.

6.1 The energy release rate and the configurational force

We start with the expression for the rate of energy dissipation, i.e., eq. (44):

$$\begin{aligned}
\Phi &= \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big[(W + K)(\mathbf{V} \cdot \mathbf{N}) + \mathbf{T}^T \dot{\mathbf{x}} \cdot \mathbf{N}\big] dS \\
&= \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big[(W - K)(\mathbf{V} \cdot \mathbf{N})\big] dS + \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \dot{\mathbf{x}}\big[\rho\dot{\mathbf{x}}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{T}\mathbf{N}\big] dS.
\end{aligned}$$

Recalling relation (9), the above equation can be written as

$$
\begin{aligned}
\Phi &= \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} [(W-K)(\mathbf{V}\cdot\mathbf{N}) - \mathbf{F}\mathbf{V}(\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N})]\, dS \\
&\qquad\qquad + \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \widetilde{\mathbf{V}}\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}]\, dS \\
&= \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \mathbf{V}\cdot\left[\left((W-K)\mathbf{I} - \mathbf{F}^T\mathbf{T}\right)\mathbf{N} - \rho\mathbf{F}^T\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})\right] dS \\
&\qquad\qquad + \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \widetilde{\mathbf{V}}\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}]\, dS.
\end{aligned} \tag{62}
$$

Due to eqs. (46) and (51), eq. (62) becomes

$$
\begin{aligned}
\Phi = \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} &\mathbf{V}\cdot[\mathbf{b}\mathbf{N} + \boldsymbol{\mathcal{P}}(\mathbf{V}\cdot\mathbf{N})]\, dS \\
&+ \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \widetilde{\mathbf{V}}\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}]\, dS.
\end{aligned} \tag{63}
$$

One can prove that under specific assumptions the second term in eq. (63) vanishes. The essential step to this end is to prove the following

Proposition: *Assume that*

$$
\int_{\partial D_\epsilon} |\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}|dS, \quad \text{is bounded as} \quad \epsilon\to 0. \tag{64}
$$

Then the following convergence holds

$$
\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \left[\widetilde{\mathbf{V}}(\mathbf{X},t) - \widetilde{\mathbf{U}}(t)\right]\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}]\, dS = 0. \tag{65}
$$

PROOF: We have

$$
\begin{aligned}
&\left|\int_{\partial D_\epsilon} \left[\widetilde{\mathbf{V}}(\mathbf{X},t) - \widetilde{\mathbf{U}}(t)\right]\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}]\ dS\right| \\
&\le \int_{\partial D_\epsilon} \sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t) - \widetilde{\mathbf{U}}(t)|\ |\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}|\ dS \\
&\le \sup_{\mathbf{X}\in\partial D_\epsilon}\left(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t) - \widetilde{\mathbf{U}}(t)|\right)\int_{\partial D_\epsilon} |\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}\mathbf{N}|dS.
\end{aligned} \tag{66}
$$

On the other hand, the condition given by eq. (10) means that

$$
\lim_{\mathbf{X}\to\mathbf{Z}(t)}(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t) - \widetilde{\mathbf{U}}(t)|) = 0.
$$

From this convergence, taking into account that the boundary ∂D_ϵ shrinks onto $\mathbf{Z}(t)$ as $\epsilon\to 0$, it is implied that the following convergence holds as well

$$
\lim_{\epsilon\to 0}\left(\sup_{\mathbf{X}\in\partial D_\epsilon}(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t) - \widetilde{\mathbf{U}}(t)|)\right) = 0.
$$

The latter jointly with the assumption (64) gives

$$\lim_{\epsilon\to 0}\left[\sup_{\mathbf{X}\in\partial D_\epsilon}\left(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)|\right)\int_{\partial D_\epsilon}|\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}|dS\right]=0,$$

which, in virtue of inequality (66), completes the proof.
Next, from eq. (65) invoking the balance of physical momentum at the crack tip i.e., eq. (39), we conclude that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{V}}(\mathbf{X},t)\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS\\&=\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{U}}(t)\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS=0.\end{aligned}\tag{67}$$

Therefore, taking into account eq. (67), the energy flux at the crack tip, i.e., eq. (63) becomes

$$\Phi=\mathbf{V}\cdot\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}(\mathbf{bN}+\mathcal{P}(\mathbf{V}\cdot\mathbf{N}))dS$$

or, due to eq. (50),

$$\Phi=-\mathbf{V}\cdot\boldsymbol{\mathcal{F}}.\tag{68}$$

Using the definition (45) for the energy release rate, eq. (68) gives the following result

$$G=-\boldsymbol{\mathcal{F}}\cdot\mathbf{t},\tag{69}$$

which confirms that the energy release rate G is the crack driving force.

Remark 5: In the preceding analysis, two relations, which can be viewed as constraints on the singularity order for the velocity and the stress fields at the crack tip, have been arisen. These relations are the condition (64) and the equation (39). Suppose that the independent variables of a function $f(\mathbf{X},t)$ (say f be the velocity or the stress tensor) can be separated as

$$f(\mathbf{X},t)=g(r)h(\theta,t),$$

where $r=|\mathbf{X}-\mathbf{Z}(t)|$ and θ are the distance and the angle, respectively, in a polar coordinate system with its origin at the crack tip. Then, assuming that $g(r)=O(r^p),\ p\geq -1$ is sufficient to assure that the condition (64) holds. Furthermore, assuming that $g(r)=O(r^p),\ p>-1$, we obtain that eq. (39) holds, as well. However, it is well known that for a linear elastic, cracked body, both the near tip velocity and stress fields are of order $O(r^{-\frac{1}{2}})$ (Freund 1981).

6.2 The rotational energy release rate and the configurational moment at the crack tip

In this section, we will show that a relationship between the rotational energy release rate and the configurational fields at the crack tip can be established. We start with the relation (57) which can be written as

$$\boldsymbol{\mathcal{M}}=-\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\mathbf{R}\times\mathbf{\mathcal{C}N}\,dS,\tag{70}$$

where

$$\mathcal{C} = \mathcal{P} \otimes \mathbf{V} + \mathbf{b}^T. \tag{71}$$

Considering that the crack evolves along an arbitrary smooth curve, we introduce the angular velocity of the crack tip

$$\boldsymbol{\omega}(t) = \omega \mathbf{m}, \tag{72}$$

where $\mathbf{m} = \mathbf{t} \times \mathbf{n}$ is the unit normal vector to the plane of the crack. Furthermore, denoting with $\mathbf{a} = a\ \mathbf{n}$ the instantaneous radius of curvature ($\mathbf{n}$ the unit normal to the curve), we can write

$$\omega = \frac{V}{a}. \tag{73}$$

Also, we denote with $\mathbf{R}_Z = \mathbf{Z} - \mathbf{0} = R_Z \mathbf{e}$ the position vector of the crack tip. Then, we can write (see Fig.4)

$$\mathbf{R}_Z = \mathbf{R} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = -\epsilon \mathbf{N}.$$

Therefore, the configurational moment $\mathcal{M}$ becomes

$$\mathcal{M} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\mathbf{R}_Z \times \mathcal{C}\mathbf{N}\big)\, dS + \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\boldsymbol{\epsilon} \times \mathcal{C}\mathbf{N}\big)\, dS. \tag{74}$$

One can prove that the last term in eq. (74) vanishes under a particular condition. Indeed,

$$\int_{\partial D_\epsilon} \big(\boldsymbol{\epsilon} \times \mathcal{C}\mathbf{N}\big)\, dS = -\int_{\partial D_\epsilon} \big(\epsilon\mathbf{N} \times \mathcal{C}\mathbf{N}\big)\, dS = -\epsilon \int_{\partial D_\epsilon} \big(\mathbf{N} \times \mathcal{C}\mathbf{N}\big)\, dS.$$

In addition, it holds

$$\Big|\int_{\partial D_\epsilon} \big(\mathbf{N} \times \mathcal{C}\mathbf{N}\big)\, dS\Big| \leq \int_{\partial D_\epsilon} \big(|\mathbf{N}| \times |\mathcal{C}\mathbf{N}|\big)\, dS = \int_{\partial D_\epsilon} |\mathcal{C}\mathbf{N}|\, dS.$$

Thus, assuming that the integral $\int_{\partial D_\epsilon} |\mathcal{C}\mathbf{N}|\, dS$ is bounded as $\epsilon \to 0$, we obtain that the integral $\int_{\partial D_\epsilon} \big(\boldsymbol{\epsilon} \times \mathcal{C}\mathbf{N}\big)\, dS$ vanishes as $\epsilon \to 0$.
Consequently, the expression for $\mathcal{M}$ (eq. (74)) becomes

$$\mathcal{M} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\mathbf{R}_Z \times \mathcal{C}\mathbf{N}\big)\, dS = -\mathbf{R}_Z \times \left(\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \mathcal{C}\mathbf{N}\, dS\right) \tag{75}$$

Notice that using the relation (71), the quantity $\mathcal{F}$ is written

$$\mathcal{F} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \mathcal{C}\mathbf{N}\, dS.$$

So, $\mathcal{M}$ is given by the following simple formula

$$\mathcal{M} = \mathbf{R}_Z \times \mathcal{F}, \tag{76}$$

which confirms the term configurational moment at the crack tip, since, essentially, it is the moment of the configurational force at the crack tip.
In addition, if θ is the angle from the $\mathbf{t}$-axis to the $\mathbf{e}$-axis, then $\mathcal{M}$ is written in terms of $\mathcal{F}$ as follows

$$\mathcal{M} = R_Z \big(\cos\theta\ (\mathcal{F} \cdot \mathbf{n}) - \sin\theta (\mathcal{F} \cdot \mathbf{t})\big)\mathbf{m}. \tag{77}$$

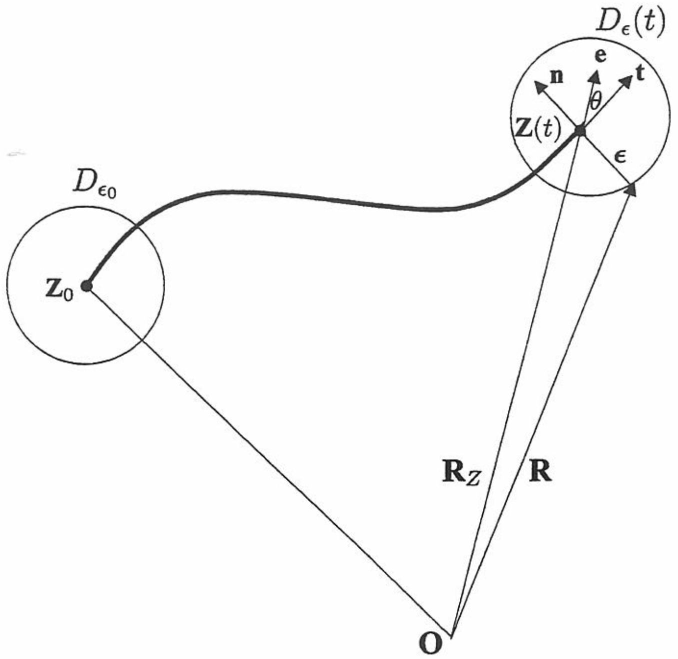

Figure 4: Some geometrical characteristics of the crack

Next, we calculate the product

$$\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}} = \omega R_Z\big(\cos\theta\ (\boldsymbol{\mathcal{F}} \cdot \mathbf{n}) - \sin\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{t})\big) \tag{78}$$

in order to take an expression for the rate of dissipation in terms of the configurational force and the configurational moment at the crack tip

$$\Phi = \frac{a}{R_Z \sin\theta}(\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}) - V\cot\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{n}). \tag{79}$$

Furthermore, if we denote with

$$G_r = \frac{\Phi}{\omega} \tag{80}$$

the *rotational energy release rate*, that is, the energy flow into the crack tip per unit angle extension of the crack, then from eq. (79) we have

$$G_r = \frac{a}{R_Z \sin\theta}(\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}) - a\cot\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{n}). \tag{81}$$

Remark 6: Assuming that $\mathbf{Z}(t)$ is a C^2 function, the instantaneous radius of curvature at $\mathbf{Z}(t)$ is related to the instantaneous curvature at $\mathbf{Z}(t)$ by $a = \frac{1}{|k|}$ and

$$|k| = \frac{\left|\frac{d\mathbf{Z}}{dt} \times \frac{d^2\mathbf{Z}}{dt^2}\right|}{\left|\frac{d\mathbf{Z}}{dt}\right|^3} = \frac{|\mathbf{V} \times \mathbf{A}|}{|\mathbf{V}|^3}, \tag{82}$$

where $\mathbf{A} = d^2\mathbf{Z}/dt^2$ is the acceleration vector of the crack tip. Therefore, we can write Φ (eq. (79)) in the following alternative form

$$\Phi = \frac{V^2}{R_Z A_n \sin\theta}(\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}) - V\cot\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{n}), \tag{83}$$

where A_n is the normal component of the vector $\mathbf{A}$.

Remark 7: In the case in which the crack is circular and the origin of the coordinates' system coincides with the center of the circle, the formulas (79) and (81) give

$$\Phi = \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}, \quad G_r = \boldsymbol{\mathcal{M}} \cdot \mathbf{m} = \mathcal{M}. \tag{84}$$

We can see from eq. $(84)_2$ that the rotational energy release rate is simply the magnitude of the vector of the configurational moment at the crack tip. Analogous results to eq. $(84)_1$ have been provided by (Maugin and Trimarco 1995) as well as by (Budiansky and Rice 1973) for disclinations and cavities, respectively. Of course, someone can easily see the analogy between the relations (68), (69) and $(84)_1$, $(84)_2$, respectively.

7 CONCLUSIONS

The objective of this paper was the study of the crack propagation within an elastic medium in the context of configurational mechanics. To this end, we proposed an appropriate kinematics and we formulated the corresponding transport and divergence theorems. In the sequel, we produced a rigorous localization process which has been used to derive the local equations for both the physical and configurational fields.

A significant consequence of the localization process was the expression for the configurational force at the crack tip related to the J–integral as well as the corresponding one for the configurational moment at the crack tip which is related to the L–integral. Based on these expressions, we derived a relationship between the configurational force at the crack tip and the energy release rate, as well as a relation connecting the rotational energy release rate with the configurational moment and force at the crack tip.

In the case of a crack with non constant curvature, the rotational energy release rate depends essentially on the geometrical characteristics of the curve. Therefore, in order to apply the formula (81), the geometrical characteristics of the curve along which the crack will evolve should be a priori known. Such situations appear in delamination cracks, where the crack necessarily follows a particular curve.

REFERENCES

Agiasofitou, E. K. and V. K. Kalpakides (2003). The concept of a balance law for a cracked elastic body and the configurational force and moment at the crack tip. Submitted for publication.

Budiansky, B. and J. R. Rice (1973). Conservation Laws and Energy-Release Rates *J. Appl. Mech. 40*, 201–203.

Chadwick, P. (1976). *Continuum Mechanics, Concise Theory and Problems*, London: George Allen & Unwin.

Dascalu, C. and G. A. Maugin (1995). The Thermoelastic Material-momentum Equation *J. Elasticity 39*, 201–212.

Eischen, J. W. and G. Herrmann (1987). Energy Release Rates and Related Balance Laws in Linear Elastic Defect Mechanics, *ASME J. Appl. Mech. 54*, 388-392.

Eshelby (1995). The elastic energy-momentum tensor. *J. Elasticity 5*, 321–326.

Fletcher, D. C. (1975). Conservation laws in linear elastodynamics. *Arch. Rat. Mech. Anal. 60*, 329–353.

Freund L. B. (1989). *Dynamic Fracture Mechanics*. Cambridge: Cambridge University Press.

Golebiewska Herrmann, A. and G. Herrmann (1981). On Energy-Release Rates of a plane Crack. *ASME J. Appl. Mech. 48*, 525–528.

Golebiewska Herrmann, A. (1982). Material Momentum Tensor and Path-Independent Integrals of Fracture Mechanics. *Int. J. Solids Stru. 18*, 319–326.

Gurtin, M. E. (1981). *An Introduction to Continuum Mechanics.* Boston: Academic Press.

Gurtin, M. E.(2000). Configurational Forces as Basic Concepts of Continuum Physics. New York: Springer.

Gurtin, M. E. and P. Podio-Guidugli (1996). *Configurational Forces and the Basic Laws for Crack Propagation. J. Mech. Phys. Solids 44*, 905–927.

Kalpakides, V. K. and E. K. Agiasofitou (2002). Configurational balance laws for dynamical fracture. *Theor. Appl. Mech. 28-29*, 205–219.

Kalpakides, V. K. and E. K. Agiasofitou (2002). On material equations in second gradient electroelasticity. *J. Elasticity 67*, 205–227.

Kienzler, R. and G. Herrmann (2000). *Mechanics in Material Space with Applications to Defect and Fracture Mechanics*. Berlin: Springer.

Knowles, J. K. and E. Sternberg (1972). On a class of conservation laws in linearized and finite elastostatics. *Arch. Rat. Mech. Anal. 44*, 187–211.

Maugin, G. A. (1992). *The Thermomechanics of Plasticity and Fracture*. Cambridge: Cambridge University Press .

Maugin, G. A. (1993). *Material Inhomogeneities in Elasticity*. London: Chapman & Hall.

Maugin, G. A. (1995). Material Forces: Concepts and applications. *ASME Appl. Mech. Rev. 48*, 213–245.

Maugin, G. A. and C. Trimarco (1995). Dissipation of configurational forces in defective elastic solids *Arch. Mech. 47*, 81–99.

Maugin, G. A. and V. K. Kalpakides (2002). A Hamiltonian formulation for elasticity and thermoelasticity *J. Phys. A: Math. Gen. 35*, 10775–10788.

Podio-Guidugli, P. (2002). Configurational forces: are they needed? *Mech. Res. Comm. 29*, 513-519.

Steinmann, P. (2000). Application of material forces to hyperelastostatic fracture mechanics. I. Continuum mechanical setting. *Int. J. Solids Stru. 37*, 7371–7391.

On the solvability of a Neumann boundary value problem

BY
M.K.Grammatikopoulos[1], P.S.Kelevedjiev[2] and N.I.Popivanov[3]
[1]Department of Mathematics, University of Ioannina, Ioannina, Greece
mgrammat@cc.uoi.gr
[2]Department of Mathematics, Technical University of Sliven, Sliven, Bulgaria
keleved@mailcity.com
[3]Faculty of Mathematics, "St. Kl. Ohridski" University of Sofia, Sofia, Bulgaria
nedyu@fmi.uni-sofia.bg

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1.Introduction

The purpose of this paper is to establish existence of $C^2[0,1]$-solutions to the scalar Neumann boundary value problem (BVP)

$$\begin{cases} f(t,x,x',x'')=0, \quad t\in[0,1], \\ x'(0)=a, \;\; x'(1)=b, \;\; a\neq b, \end{cases} \tag{N}$$

where the function $f(t,x,p,q)$ and its first derivatives are continuous only on suitable subsets of the set $[0,1]\times R^3$.

The solvability of the homogeneous Neumann problem for the equation $\big(p(t)x'\big)'+f(t,x,x',x'')=y(t)$ has been studied in [5,9,11]. Results, concerning the existence of solutions to the homogeneous and nonhomogeneous Neumann problem for the equation $x''=f(t,x,x',x'')-y(t)$ can be found in [5,10] and [7] respectively. BVPs for the same equation with various linear boundary conditions have been studied in [1,2,7,10]. The results of [12] guarantee the solvability of BVPs for the equation $x''=f(t,x,x',x'')$ with fully linear boundary conditions. BVPs for the equation $f(t,x,x',x'')=0$ with fully nonlinear boundary conditions have been studied in [6]. For results, which guarantee the solvability of the Dirichlet BVP for the same equation, in the scalar and in the vector cases, see [3] and [8] respectively.

Concerning the kind of the nonlinearity of the function $f(t,x,p,q)$, we note that it is assumed semilinear in [1], linear with respect to x, p and q in [2,11] and sublinear in [5], while in [11] f is a Caratheodory function. Finally, in [10] and [12] f is a linear function with respect to q, while with respect to p it is a quadratic function or satisfies Nagumo type growth conditions respectively.

As in [4,6], we use sign conditions to establish a priori bounds for x, x' and x'', where $x(t)\in C^2[0,1]$ is a solution to a suitable family of BVPs containing the problem (N). Using these a priori bounds and applying the topological transversality theorem from [4], we prove our main existence result.

2. BASIC HYPOTHESES

Our results rely on the following three hypotheses.

H1. There are constants $K_x > 0$ and $K_q > 0$ such that

$$f_x(t,x,p,q) \geq K_x \quad \text{for} \quad (t,x,p,q) \in [0,1] \times R \times J_p \times R,$$

$$f_q(t,x,p,q) \leq -K_q \quad \text{for} \quad (t,x,p,q) \in [0,1] \times J_x \times J_p \times R,$$

where $J_x = \left[\min\{0, \frac{a+b}{2}, \frac{a^2}{2(a-b)}\}, \max\{0, \frac{a+b}{2}, \frac{a^2}{2(a-b)}\}\right]$ and $J_p = [\min\{a,b\}, \max\{a,b\}]$.

H2. There are constants $K > 0$, $M > 0$ and a sufficiently small $\varepsilon > 0$ such that

$$f(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (-\infty, -M),$$

and

$$f(t,x,p,q) + Kq \leq 0 \text{ for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (M, \infty),$$

where

$$M_0 = \max\Big\{\frac{e}{e^2-1}(|a-be| + |ae-b|), \frac{Q}{\min\{K, K_q, K_x\}} + \max\{\frac{|a+b|}{2}, \frac{a^2}{2|a-b|}\}\Big\}, \tag{2.1}$$

$Q = \max\left|\lambda f\big(t,x,p,b-a-(1-\lambda)x\big) - (1-\lambda)K\big(b-a-(1-\lambda)x\big)\right|$ for $(\lambda,t,x,p) \in [0,1] \times [0,1] \times J_x \times J_p$, and the constants K_x and K_q as well as the sets J_x and J_p are as in **H1**.

H3. $f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous and $f_q(t,x,p,q) < 0$ for $(t,x,p,q) \in [0,1] \times [-M_0-\varepsilon, M_0+\varepsilon] \times [-M_1-\varepsilon, M_1+\varepsilon] \times [-M_2-\varepsilon, M_2+\varepsilon]$, where $M_1 = |a| + M_0 + M$, $M_2 = M_0 + M$, and M_0 and M are as in **H2**.

3. AUXILIARY LEMMAS

In order to obtain our main existence result, we consider the following family of BVPs

$$\begin{cases} K\big(x'' - (1-\lambda)x\big) = \lambda\Big(K\big(x''-(1-\lambda)x\big) + f\big(t,x,x',(x''-(1-\lambda)x)\big)\Big), \\ x'(0) = a, \quad x'(1) = b, \end{cases} \tag{3.1$_\lambda$}$$

where $\lambda \in [0,1]$, while $K > 0$ is as in **H2**, when **H2** holds, and prove the following two auxiliary lemmas.

Lemma 3.1. Let **H1** be hold and $x(t) \in C^2[0,1]$ be a solution to $(3.1)_\lambda$, $\lambda \in [0,1]$, where $K > 0$ is an arbitrary constant. Then

$$|x(t)| \leq M_0, \; t \in [0,1],$$

where M_0 is defined by (2.1).

Proof. For $\lambda = 0$, the problem $(3.1)_0$ is of the form

$$x'' - x = 0, \quad x'(0) = a, \quad x'(1) = b.$$

The unique solution to this BVP satisfies the bound

$$|x(t)| \leq \frac{e}{e^2-1}(|a-be| + |ae-b|), \quad t \in [0,1].$$

Let now $\lambda \in (0,1]$. Then the function $y(t) = x(t) - s(t)$, $t \in [0,1]$, where $s(t) = \frac{b-a}{2}t^2 + at$, $t \in [0,1]$, is a solution to the homogeneous boundary value problem

$$K\Big(y''+b-a-(1-\lambda)(y+s)\Big) = \lambda\Bigg(K\Big(y''+b-a-(1-\lambda)(y+s)\Big)+f\Big(t,y+s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)\Bigg),$$

$$y'(0) = y'(1) = 0.$$

From this equation we obtain

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky - (1-\lambda)K(b-a-(1-\lambda)s) + \lambda f\Big(t,y+s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky - (1-\lambda)K(b-a-(1-\lambda)s) + \lambda f\Big(t,y+s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big) -$$
$$-\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big) + \lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky - (1-\lambda)K(b-a-(1-\lambda)s) + \lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)y +$$
$$+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big) - \lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)s\Big) +$$
$$+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)s\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky - (1-\lambda)K(b-a-(1-\lambda)s) + \lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)y +$$
$$-\lambda f_q\Big(t,s,y'+s',y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\Big)(1-\lambda)y +$$
$$+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)s\Big) - \lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big) + \lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky - (1-\lambda)K(b-a-(1-\lambda)s) + \lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)y +$$
$$-\lambda(1-\lambda)f_q\Big(t,s,y'+s',y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\Big)y +$$
$$+\lambda f_q\Big(t,s,y'+s',b-a-(1-\lambda)s+\theta_3 y''\Big)y'' + \lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big),$$

$$\left\{\begin{aligned} &\Bigg((1-\lambda)K - \lambda f_q\Big(t,s,y'+s',b-a-(1-\lambda)s+\theta_3 y''\Big)\Bigg)y'' = \\ &\Bigg((1-\lambda)^2K + \lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big) - \\ &-\lambda(1-\lambda)f_q\Big(t,s,y'+s',y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\Big)\Bigg)y + \\ &+\lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big) - (1-\lambda)K(b-a-(1-\lambda)s), \end{aligned}\right. \tag{3.2}$$

where $0 < \theta_i < 1$, $i = 1,2,3$.

Next, suppose that $|y(t)|$ achieves its maximum at $t_0 \in (0,1)$. Then the function $z = y^2(t)$ has also a maximum at t_0. Consequently, we see that

$$0 \geq z''(t_0) = 2y(t_0)y''(t_0). \tag{3.3}$$

Using the fact that $y'(t_0)=0$, from (3.2) we obtain

$$\begin{cases} \Big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_3 y_0''\big)\Big) y_0'' = \\ \Big((1-\lambda)\big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', y_0''+b-a-(1-\lambda)s_0-\theta_2(1-\lambda)y_0\big)\big)+ \\ \lambda f_x\big(t_0, s_0+\theta_1 y_0, s_0', y_0''+b-a-(1-\lambda)(y_0+s_0)\big)\Big) y_0 + \\ +\lambda f\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0\big) - (1-\lambda)K(b-a-(1-\lambda)s_0), \end{cases} \tag{3.4}$$

where $s_0=s(t_0)$, $s_0'=s'(t_0)$, $y_0=y(t_0)$, $y_0''=y''(t_0)$.

On the other hand, in view of **H1**, we have

$$\begin{cases} (1-\lambda)\big((1-\lambda)K-\lambda\overline{f}_q\big)+\lambda\overline{f}_x \geq \min\{(1-\lambda)K-\lambda\overline{f}_q, \overline{f}_x\} \geq \\ \min\{K, -\overline{f}_q, \overline{f}_x\} \geq \min\{K, K_q, K_x\}, \end{cases} \tag{3.5}$$

where

$$\overline{f}_q = f_q\big(t_0, s_0, s_0', y_0''+b-a-(1-\lambda)s_0-\theta_2(1-\lambda)y_0\big),$$
$$\overline{f}_x = f_x\big(t_0, s_0+\theta_1 y_0, s_0', y_0''+b-a-(1-\lambda)(y_0+s_0)\big).$$

Suppose now that $|y(t_0)| > \frac{Q}{\min\{K,K_x,K_q\}}$. Then, from (3.4) and (3.5) it follows that

$$\begin{cases} \big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_3 y_0''\big)\big) y_0'' \geq \min\{K, K_q, K_x\} y(t_0) + \\ +\lambda f\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0\big) - (1-\lambda)K(b-a-(1-\lambda)s_0) \end{cases} \tag{3.6}$$

if $y(t_0) > \frac{Q}{\min\{K,K_x,K_q\}}$ and

$$\begin{cases} \big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_3 y_0''\big)\big) y_0'' \leq \min\{K, K_q, K_x\} y(t_0) + \\ +\lambda f\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0\big) - (1-\lambda)K(b-a-(1-\lambda)s_0) \end{cases} \tag{3.7}$$

if $y(t_0) < -\frac{Q}{\min\{K,K_x,K_q\}}$. Multiplying (3.6) and (3.7) by $y(t_0)$, we obtain

$$\big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_3 y_0''\big)\big) y_0'' y_0 \geq y_0\big(\min\{K, K_q, K_x\} y_0 - Q\big) > 0,$$

$$\big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_3 y_0''\big)\big) y_0'' y_0 \geq y_0\big(\min\{K, K_q, K_x\} y_0 + Q\big) > 0.$$

respectively. Finally, since $f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_3 y_0''\big) < 0$, we conclude that

$$y_0'' y_0 > 0,$$

which contradicts (3.3). Thus, we infer that if $|y(t)|$ achieves its maximum in $(0,1)$, then

$$|y(t)| \leq \frac{Q}{\min\{K, K_x, K_q\}} \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad \lambda \in (0,1].$$

Let $|y(1)|$ be the maximum of $|y(t)|$ and suppose that $|y(1)| > \frac{Q}{\min\{K,K_x,K_q\}}$. Following the above reasoning and the fact that $y'(1)=0$, we obtain

$$y(1)y''(1) > 0.$$

If $y(1) > 0$, then $y''(1) > 0$ and so $y'(t)$ must be a strictly increasing function for $t \in U_1$, where $U_1 \subset [0,1]$ is a sufficiently small neighbourhood of $t = 1$. So, we see that

$$y'(t) < y'(1) = 0 \quad \text{for} \quad t \in U_1 \setminus \{1\},$$

i.e. $y(t)$ is a strictly decreasing function for $t \in U_1$. Therefore, $y(1) = |y(1)|$ can not be the maximum of $|y(t)|$ on $[0,1]$, which is a contradiction. Assume next that $y(1) < 0$. Then a similar to the above arguments lead again to a contradiction. Thus, we see that

$$|y(1)| \leq \frac{Q}{\min\{K, K_x, K_q\}}.$$

The inequality

$$|y(0)| \leq \frac{Q}{\min\{K, K_x, K_q\}}$$

can be obtained in the same manner. Consequently, the solutions of $(3.1)_\lambda$, $\lambda \in (0,1]$, satisfy the bound

$$|x(t)| \leq \frac{Q}{\min\{K, K_x, K_q\}} + \max\Big\{\frac{a^2}{2|a-b|}, \frac{|a+b|}{2}\Big\}, \quad t \in [0,1],$$

and the proof of the lemma is complete.□

LEMMA 3.2. Let **H1** and **H2** be hold and let $x(t) \in C^2[0,1]$ be a solution to $(3.1)_\lambda$, $\lambda \in [0,1]$, where K is as in **H2**. Then:

(a)

$$|x''(t) - (1-\lambda)x(t)| \leq M, \quad |x''(t)| \leq M_2, \quad t \in [0,1],$$

where $M_2 = M_0 + M$;

(b)

$$|x'(t)| \leq M_1, t \in [0,1],$$

where $M_1 = |a| + M_0 + M$.

Proof. (a) Suppose that there exists a $(t_0, \lambda_0) \in [0,1] \times [0,1]$ or a $(t_1, \lambda_1) \in [0,1] \times [0,1]$ such that

$$x''(t_0) - (1-\lambda_0)x(t_0) < -M \quad \text{or} \quad x''(t_1) - (1-\lambda_1)x(t_1) > M.$$

By Lemma 3.1, we have

$$|x(t)| \leq M_0 \text{ for } t \in [0,1]. \tag{3.8}$$

In particular, (3.8) holds for t_0 or t_1. Thus, in view of **H2**, we have

$$0 > K\Big(x''(t_0) - (1-\lambda_0)x(t_0)\Big) = \lambda_0\bigg(K\Big(x''(t_0) - (1-\lambda_0)x(t_0)\Big) +$$

$$+ f\Big(t_0, x(t_0), x'(t_0), x''(t_0) - (1-\lambda_0)x(t_0)\Big)\bigg) \geq 0$$

or

$$0 < K\Big(x''(t_1) - (1-\lambda_1)x(t_1)\Big) = \lambda_1\bigg(K\Big(x''(t_1) - (1-\lambda_1)x(t_1)\Big) +$$

$$+ f\Big(t_1, x(t_1), x'(t_1), x''(t_1) - (1-\lambda_1)x(t_1)\Big)\bigg) \leq 0,$$

respectively, which is a contradiction. The obtained contradiction shows that

$$-M \leq x''(t) - (1-\lambda)x(t) \leq M \text{ for } t \in [0,1] \text{ and } \lambda \in [0,1],$$

and therefore

$$-(M_0+M) \le x''(t) \le M_0+M \quad \text{for} \quad t\in[0,1],$$

which proves (a).

(b) Observe that, by the mean value theorem, for each $t\in(0,1]$ there is a $\xi\in(0,t)$ such that

$$x'(t)-x'(0)=x''(\xi)t.$$

Since, in view of (a), we have $|x''(\xi)|\le M_0+M$, from the last formula we find that

$$|x'(t)|\le|x'(0)|+|x''(\xi)|\le|a|+M_0+M,\ t\in[0,1],$$

which proves (b) and completes the proof of the lemma.□

4.THE MAIN RESULT

Our main result is the following existence theorem, the proof of which is based on the lemmas of the previous section and the topological transversality theorem from [4].

THEOREM 4.1. Let **H1**, **H2** and **H3** be hold. Then the problem (N) has at least one solution in $C^2[0,1]$.

Proof. For any $(\lambda,t,x,p,q)\in[0,1]\times[0,1]\times[-M_0-\varepsilon,M_0+\varepsilon]\times[-M_1-\varepsilon,M_1+\varepsilon]\times[-M_2-\varepsilon,M_2+\varepsilon]$ consider the function $h(\lambda,t,x,p,q)=\lambda\big(Kq+f(t,x,p,q)\big)-Kq$, where M_i, $i=0,1,2$ are the constants for which, in view of Lemmas 3.1 and 3.2, each $C^2[0,1]$-solution $x(t)$ to $(3.1)_\lambda$, $\lambda\in[0,1]$, satisfies the bounds

$$|x(t)|\le M_0,\quad |x'(t)|\le M_1, |x''(t)-(1-\lambda)x(t)|\le M,\quad \text{and}\quad |x''(t)|\le M_2,\ \text{for}\ t\in[0,1], \tag{3.9}$$

respectively. Since $M_2>M$, in view of **H2**, we obtain

$$h(\lambda,t,x,p,-M_2-\varepsilon)>0 \quad\text{and}\quad h(\lambda,t,x,p,M_2+\varepsilon)<0$$

for $(\lambda,t,x,p)\in[0,1]\times[0,1]\times[-M_0-\varepsilon,M_0+\varepsilon]\times[-M_1-\varepsilon,M_1+\varepsilon]$. Besides, by **H3,** we see that $h(\lambda,t,x,p,q)$ and $h_q(\lambda,t,x,p,q)$ are continuous functions and $h_q(\lambda,t,x,p,q)<0$ for $(\lambda,t,x,p,q)\in[0,1]\times[0,1]\times[-M_0-\varepsilon,M_0+\varepsilon]\times[-M_1-\varepsilon,M_1+\varepsilon]\times[-M_2-\varepsilon,M_2+\varepsilon]$. Therefore, there is a unique function $G(\lambda,t,x,p)$, which is continuous on the set $[0,1]\times[0,1]\times[-M_0-\varepsilon,M_0+\varepsilon]\times[-M_1-\varepsilon,M_1+\varepsilon]$ and such that

$$q=G(\lambda,t,x,p),\ \ (\lambda,t,x,p)\in[0,1]\times[0,1]\times[-M_0-\varepsilon,M_0+\varepsilon]\times[-M_1-\varepsilon,M_1+\varepsilon],$$

is equivalent to the equation

$$h(\lambda,t,x,p,q)=0,\ (\lambda,t,x,p,q)\in[0,1]\times[0,1]\times[-M_0-\varepsilon,M_0+\varepsilon]\times[-M_1-\varepsilon,M_1+\varepsilon]\times[-M_2-\varepsilon,M_2+\varepsilon].$$

So, since $|x''(t)-(1-\lambda)x(t)|\le M<M_2+\varepsilon$ for $t\in[0,1]$ and $\lambda\in[0,1]$, the family $(3.1)_\lambda$ is equivalent to the following families of BVPs

$$\begin{cases} x''-(1-\lambda)x=G(\lambda,t,x,x'), & t\in[0,1],\\ x'(0)=a,\ x'(1)=b, \end{cases} \tag*{$(3.10)_\lambda$}$$

and

$$\begin{cases} x''-(2-\lambda)x=G(\lambda,t,x,x')-x, & t\in[0,1],\\ x'(0)=a,\ x'(1)=b, \end{cases} \tag*{$(3.11)_\lambda$}$$

$\lambda \in [0,1]$. Note that from $h(0,t,x,p,0) = 0$ it follows that

$$G(0,t,x,p) = 0 \text{ for } (t,x,p) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon]. \tag{3.12}$$

Now, for $C^2_B[0,1] = \{x(t) \in C^2[0,1] : x'(0) = a,\ x'(1) = b\}$ define the set

$$U = \left\{x \in C^2_B[0,1] : |x| < M_0 + \varepsilon,\ |x'| < M_1 + \varepsilon,\ |x''| < M_2 + \varepsilon\right\}$$

and then for $\lambda \in [0,1]$ define the maps

$$G_\lambda : C^1[0,1] \to C[0,1] \quad \text{by} \quad (G_\lambda x)(t) = G\big(\lambda, t, x(t), x'(t)\big) - x(t),\ t \in [0,1],$$

$$j : C^2_B[0,1] \to C^1[0,1] \quad \text{by} \quad jx = x \quad \text{and} \quad L_\lambda : C^2_B[0,1] \to C[0,1] \quad \text{by} \quad L_\lambda x = x'' - (2-\lambda)x.$$

Since L_λ, $\lambda \in [0,1]$, is a continuous, linear, one-to-one map of $C^2_B[0,1]$ onto $C[0,1]$, the map L_λ^{-1}, $\lambda \in [0,1]$, exists and is continuous. In addition, G_λ, $\lambda \in [0,1]$, is a continuous and j is a completely continuous embedding. Since $j(\overline{U})$ is a compact subset of $C^1[0,1]$, and G_λ, $\lambda \in [0,1]$, and L_λ^{-1}, $\lambda \in [0,1]$, are continuous on $j(\overline{U})$ and $G_\lambda(j(\overline{U}))$ respectively, the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1] \text{ defined by } H(x,\lambda) \equiv H_\lambda(x) \equiv L_\lambda^{-1} G_\lambda j(x)$$

is compact. Besides, the equation

$$L_\lambda^{-1} G_\lambda j(x) = x \quad \text{for } x \in \overline{U} \quad \text{yields} \quad L_\lambda x = G_\lambda j x,$$

coincides with the BVP $(3.11)_\lambda$. Thus, the fixed points of $H_\lambda(x)$ are solutions to $(3.11)_\lambda$. But, by (3.9), the solutions to $(3.11)_\lambda$ are elements of U. Consequently, $H_\lambda(x), \lambda \in [0,1]$, is a fixed point free on ∂U, i.e. $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Finally, using (3.12), we see that the map H_0 is a constant map, i.e. $H_0(x) \equiv l$, where l is the unique solution to the BVP

$$x'' - 2x = -x, \quad x'(0) = a, \quad x'(1) = b.$$

From the fact that $l \in U$ it follows that H_0 is an essential map (see, [4]). By the topological transversality theorem (see, [4]), $H_1 = L_1^{-1} G_1 j$ is also essential. So, the problem $(3.11)_1$ has a $C^2[0,1]$-solution. That is, $(3.10)_1$ has a $C^2[0,1]$-solution. To complete the proof, remark that the problem $(3.10)_1$ is equivalent to $(3.1)_1$, which coincides with the problem (N).□

We conclude with the following example, which illustrates our main result.

EXAMPLE 4.1. Consider the boundary value problem

$$1 - (1.5 - t)x'' - t x''^5 - \cos x' + x = 0,$$

$$x'(0) = 0, \quad x'(1) = 10^{-4}.$$

Clearly, **H1** holds for $K_x = 1$, $K_q = 0.5$, $J_x = [0, 5.10^{-5}]$ and $J_p = [0, 10^{-4}]$. Next, observe that

$$5.10^{-5} \le 10^{-4} - (1-\lambda)x \le 10^{-4} \quad \text{for} \quad x \in J_x$$

and choose $K = 0.5$. Then, from

$$-1,5.10^{-4} - 10^{-20} \le -(1,5-t)\big(10^{-4} - (1-\lambda)x\big) - t\big(10^{-4} - (1-\lambda)x\big)^5 \le -2,5.10^{-5}$$

for $(\lambda, t, x) \in [0,1] \times [0,1] \times J_x$ and

$$0 \le 1 - \cos p \le 5.10^{-9} \text{ for } p \in J_p$$

it follows that

$$-16.10^{-5} \leq 1-(1,5-t)\Big(10^{-4}-(1-\lambda)x\Big)-t\Big(10^{-4}-(1-\lambda)x\Big)^5-\cos p+x \leq 25.10^{-6}+5.10^{-9}$$

for $(\lambda,t,x,p) \in [0,1]\times[0,1]\times J_x \times J_p$. Therefore $Q=\max\{16.10^{-5},\, 0,5.10^{-4}\}=16.10^{-5}$. Note that

$$M_0=\max\Big\{\frac{e}{e^2-1}(|10^{-4}e|+|10^{-4}|),\frac{16.10^{-5}}{\min\{1,\frac{1}{2}\}}+5.10^{-5}\Big\}=37.10^{-5}$$

and, as it is easy to see, **H2** and **H3** hold for $M=5$ and $\varepsilon=3.10^{-5}$. Thus, we can apply Theorem 4.1 to conclude that the considered problem has a solution in $C^2[0,1]$.

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References

[1] P.M.Fitzpatrick, *Existence results for equations involving noncompact perturbation of Fredholm mappings with applications to differential equations,* J. Math. Anal. Appl. 66 (1978), 151-177.

[2] P.M.Fitzpatrick,W.V. Petryshyn, *Galerkin method in the constructive solvability of nonlinear Hammerstein equations with applications to differential equations,* Trans. Amer. Math. Soc. 238 (1978), 321-340.

[3] M.K.Grammatikopoulos, P.S.Kelevedjiev, *Minimal and maximal solutions for two-point boundary value problems,* Electron. J. Diff. Eqns. 21 (2003), 1-14.

[4] A.Granas, R.B.Guether, J.W.Lee, *Nonlinear boundary value problems for ordinary differential equations,* Dissnes Math., Warszawa, 1985.

[5] G.Hetzer, V.Stallbohm, *Eine Existenzaussage für asymptotisch lineare Störungen eines Fredholmoperators mit Index 0,* Manuscr. Math. 21 (1977), 81-100. New York, Dekker, 1994.

[6] P.Kelevedjiev, N.Popivanov, *Existence of solutions of boundary value problems for the equation $f(t,x,x',x'')=0$ with fully nonlinear boundary conditions,* Annuaire de l'Universite de Sofia 94, 2000, 65-77.

[7] Y.Mao, J.Lee, *Two point boundary value problems for nonlinear differential equations,* Rocky Maunt. J. Math. 26 (1996), 1499-1515.

[8] S.A.Marano, *On a boundary value problem for the differential equation $f(t,x,x',x'')=0$,* J. Math. Anal. Appl. 182 (1994), 309-319.

[9] W.V.Petryshyn, *Fredholm theory for abstract and differential equations with noncompact nonlinear perturbations of Fredholm maps,* J. Math. Anal. Appl. 72 (1979), 472-499.

[10] W.V.Petryshyn, *Solvability of various boundary value problems for the equation $x''=f(t,x,x',x'')-y$,* Pacific J. Math. 122 (1986), 169-195.

[11] W.V.Petryshyn, Z.S.Yu, *Solvability of Neumann BV problems for nonlinear second order ODE's which need not be solvable for the highest order derivative,* J. Math. Anal. Appl. 91 (1983), 244-253.

[12] W.V.Petryshyn, Z.S.Yu, *Periodic solutions of nonlinear second-order differential equations which are not solvable for the highest-order derivative,* J. Math. Anal. Appl. 89 (1982), 462-488.

[13] A.Tineo, *Existence of solutions for a class of boundary value problems for the equation* $x'' = F(t, x, x', x'')$, Comment. Math. Univ. Carolin 29 (1988), 285-291.